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Soliton surface associated with the WDVV equation for $n = 3$ case

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Abstract. This paper describes the soliton surfaces approach to the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) equation. We constructed the surface associated with the WDVV equations using Sym-Tafel formula, which gives a connection between the classical geometry of manifolds immersed in $\mathbf{R}^m$ and the theory of solitons. The so-called Sym-Tafel formula simplifies the explicit reconstruction of the surface from the knowledge of its fundamental forms, unifies various integrable nonlinearities and enables one to apply powerful methods of the soliton theory to geometrical problems. The soliton surfaces approach is very useful in construction of the so-called integrable geometries. Indeed, any class of soliton surfaces is integrable. Geometrical objects associated with soliton surfaces (tangent vectors, normal vectors, foliations by curves etc.) usually can be identified with solutions to some nonlinear models (spins, chiral models, strings, vortices etc.) [1], [2]. We consider the geometry of surfaces immersed in Euclidean spaces. Such soliton surfaces for the WDVV equation for $n = 3$ case with an antidiagonal metric $\eta_{11} = 0$ are considered, and first and second fundamental forms of soliton surfaces are found for this case. Also, we study an area of surfaces for the WDVV equation for $n = 3$ case with an antidiagonal metric $\eta_{11} = 0$.

1. Introduction

In this paper we shall consider so-called nonlinear partial differential equations of associativity in 2D topological field theories (see [3]-[6]) and give their description as integrable nondiagonalizable weakly nonlinear systems of hydrodynamic type. For systems of this type corresponding general differential geometric theory of integrability connected with Poisson structures of hydrodynamic type can be developed. We remind very briefly following Dubrovin [3] the basic mathematical concepts connected with the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) system arising originally in two-dimensional topological field theories [3], [4] and its relations with the Dubrovin type equations of associativity. The WDVV equations, in general, have the following form [3]:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p} \eta^{pq} \frac{\partial^3 F}{\partial t^q \partial t^k \partial t^r} = \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^p} \eta^{pq} \frac{\partial^3 F}{\partial t^i \partial t^q \partial t^r}, \quad \forall i, j, k, r \in \{1, ..., n\},$$

where F is a prepotential, η is a metric.

Consider a function $F(t)$, $t = (t^1, ..., t^n)$ such that the following three conditions are satisfied for its third derivatives denoted as [3], [4]

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} :$$

1) normalization, i.e.,

$$\eta_{\alpha\beta} = c_{1\alpha\beta}(t)$$

is a constant nondegenerate matrix;

2) associativity, i.e., the functions

$$c^\gamma_{\alpha\beta}(t) = \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t)$$

for any t define a structure of an associative algebra A_t in the n-dimensional space with a basis $e_1, ..., e_n$:

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma$$

3) $F(t)$ must be a quasihomogeneous function of its variables:

$$F(c^{d_1} t^1, ..., c^{d_n} t^n) = c^{d_F} F(t^1, ..., t^n)$$

for any nonzero c and for some numbers $d_1, ..., d_n, d_F$.

The resulting system of equations for $F(t)$ is called the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) system [5], [6] (see also [[3], [4]]). It was shown by Dubrovin [3] that solutions of the WDVV system can be reduced by a linear change of coordinates to two special types:

(1) in physically the most important case

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, ..., t^n)$$

for some function $f(t^2, ..., t^n)$

(2) in some special case

$$F(t) = \frac{1}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=1}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, ..., t^n).$$

In this work we consider the WDVV equations for $n = 3$ case with an antidiagonal metric such that $\eta_{11} = 0$

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this case, the dependence of the function F on the fixed variable t^1 was found by Dubrovin [7], [8] which is

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3)$$

For this case the equation of associativity reduces to the following nonlinear equation of the third order for a function $f = f(x, t))$ of two independent variables ($x = t^2, t = t^3$):

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt} \tag{1}$$

.

Let us introduce new variables a, b, c as follows [8], [9]:

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

In the above variables the equation (1) can be rewritten as a system of three equations in the following way:

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \tag{2}$$

.

In the following sections we work with the system (2).

2. Soliton surfaces for WDVV equation for $n = 3$
2.1. First fundamental form of a surface
The corresponding Lax pair for the WDVV equation for $n = 3$ case to the system (2) is given by

$$\Phi_x = U\Phi \tag{3}$$
$$\Phi_t = V\Phi \tag{4}$$

where $U = \lambda A$ and $V = \lambda B$. Here A and B matrices defined as follows [8], [9]:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2 - ac & c & 0 \end{pmatrix}. \tag{5}$$

The scalar square of the total differential $d\mathbf{r}$ of the radius-vector of the current point of a surface is called the first fundamental form I of the surface [10]:

$$I = d\mathbf{r}^2,$$

In expanded form, it is recorded as

$$I = \mathbf{r}_x^2 dx^2 + 2\mathbf{r_x}\mathbf{r_t}dxdt + \mathbf{r}_t^2 dt^2, \tag{6}$$

where x and t are the curvatures.

To construct the surface, we now use the Sym-Tafel formula [11]. It has the form

$$r = \Phi^{-1}\Phi_\lambda,$$

where $r = \sum r_j\sigma_j$ is the matrix form of the position vector of the surface, Φ is a solution of the equations (3)-(4). We have

$$r_x = \Phi^{-1}U_\lambda\Phi, \quad r_t = \Phi^{-1}V_\lambda\Phi.$$

In terms of the Lax representation, equation (6) will be rewritten as follows:

$$I = \frac{1}{2}\left(\text{tr}(U_\lambda^2)dx^2 + 2\,\text{tr}(U_\lambda V_\lambda)dxdt + \text{tr}(V_\lambda^2)dt^2\right). \tag{7}$$

We now turn to finding the first fundamental form of soliton surface for the WDVV equation for n = 3 case to the system (4)

$$\text{tr}(U_\lambda^2) = a^2 + 4b, \tag{8}$$

$$\text{tr}(U_\lambda V_\lambda) = ab + 3c, \tag{9}$$

$$\text{tr}(V_\lambda^2) = 3b^2 - 2ac. \tag{10}$$

Substituting equations (8)-(10) into equation (7) we have the first fundamental form of soliton surface for the WDVV equation to the system (2)

$$I = \frac{1}{2}\left[(a^2 + 4b)\mathrm{d}x^2 + 2(ab + 3c)\mathrm{d}x\mathrm{d}t + (3b^2 - 2ac)\mathrm{d}t^2\right]$$

2.2. Second fundamental form of a surface

The scalar product of the total differential of the second order $\mathrm{d}^2\mathbf{r}$ of the radius-vector $\mathbf{r}$ of the current point of a surface by the orbit of the normal $\mathbf{n}$ at this point is called the second quadratic form of the surface [10]:

$$II = -\mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{r},$$

where

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_t}{|\mathbf{r}_x \wedge \mathbf{r}_t|}.$$

In an expanded form, it is recorded as

$$II = b_{11}dx^2 + 2b_{12}dxdt + b_{22}dt^2,$$

where the coefficients b_{11}, b_{12} and b_{22} are given as

$$b_{11} = \mathbf{r}_{xx} \cdot \mathbf{n},$$
$$b_{12} = \mathbf{r}_{xt} \cdot \mathbf{n},$$
$$b_{22} = \mathbf{r}_{tt} \cdot \mathbf{n},$$

or

$$b_{11} = \frac{1}{2}\text{tr}(r_{xx}n), \tag{11}$$

$$b_{12} = \frac{1}{2}\text{tr}(r_{xt}n), \tag{12}$$

$$b_{22} = \frac{1}{2}\text{tr}(r_{tt}n), \tag{13}$$

where

$$r_{xx} = \Phi^{-1}(U_{\lambda x} + [U_\lambda, U])\Phi,$$
$$r_{xt} = \Phi^{-1}(U_{\lambda t} + [U_\lambda, V])\Phi,$$
$$r_{tt} = \Phi^{-1}(V_{\lambda t} + [V_\lambda, V])\Phi$$

The normal vector n is given by

$$n = \pm\frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2}\text{tr}([U_\lambda, V_\lambda]^2)}}.$$

Thus, the equation (11)-(13) is written as follows

$$b_{11} = \frac{1}{2} \frac{\operatorname{tr}\left((U_{\lambda x} + [U_\lambda, U])[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{14}$$

$$b_{12} = \frac{1}{2} \frac{\operatorname{tr}\left((U_{\lambda t} + [U_\lambda, V])[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{15}$$

$$b_{22} = \frac{1}{2} \frac{\operatorname{tr}\left((V_{\lambda t} + [V_\lambda, V])[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{16}$$

Using equation (5) we obtain that $[A, B] = 0$. So, we have that $n = 0$ and the second fundamental form of a soliton surface for the WDVV equation to the system (2) is

$$II = 0$$

3. Area of surfaces for WDVV equation for $n = 3$

In this section we consider the area of surfaces for the WDVV equation for $n = 3$ to the system (2). Area of surfaces is evaluated by

$$S = \iint \sqrt{\frac{1}{2}\operatorname{tr}\left(\{U_{\lambda x} + [U_\lambda, U]\}^2\right)}\,\mathrm{d}x\mathrm{d}t \tag{17}$$

where the matrix A is defined as in equation (5).
So, that $[U_\lambda, U] = 0$, we have

$$(U_{\lambda x})^2 = \begin{pmatrix} 0 & 0 & 0 \\ a_x b_x & a_x^2 & 0 \\ b_x^2 & a_x b_x & 0 \end{pmatrix}$$

Area of surfaces (17) for the WDVV equation to the system (2) is given by

$$S = \iint \sqrt{\frac{1}{2}a_x^2}\,\mathrm{d}x\mathrm{d}t = \sqrt{\frac{1}{2}}\int a\,\mathrm{d}t + k$$

where k is a constant.

4. Conclusion

In this work we considered the WDVV equations for $n = 3$ case with an antidiagonal metric $\eta_{11} = 0$. Soliton surfaces for the WDVV equations for $n = 3$ cases with an antidiagonal metric $\eta_{11} = 0$ was obtained. Area of surfaces for the WDVV equations for $n = 3$ cases with an antidiagonal metric $\eta_{11} = 0$ was investigated.

5. References

[1] A. Sym, Soliton Surfaces II 1983 Lett. Nuovo Cimento **36** 307-312.
[2] Sym A, Soliton 1985 Surfaces and their Applications in Geometrical Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Physics Vol. 239 ed. Martini R Springer- Berlin 154-231.
[3] Dubrovin B.A. 1996 *Geometry of 2D topological field theories*, in: Springer Lecture Notes in Math. **1620** 120-348.
[4] B.A.Dubrovin 1992 Integrable systems in topological field theory, Nucl. Physics B **379** 627-689

[5] E.Witten 1990 On the structure of the topological phase of two-dimensional gravity Nucl. Physics B **340** 281-332

[6] R.Dijkgraaf, E.Verlinde and H.Verlinde 1991 Nucl. Physics B, 352 59; November 1990 Notes on topological string theory and 2D quantum gravity Preprint PUPT-1217 IASSNS-HEP-90/80

[7] Mokhov O.I., *Realization of Frobenius manifolds as submanifolds in pseudo-Euclidean spaces*, arXiv:0911.4212.

[8] Mokhov O.I., Ferapontov Y.V. 1996 *The associativity equations for two-dimensional topological field theory as integrable hamiltonian non-diagonalizable systems of hydrodynamic type* Functional analysis and its applications **30 (3)** 62-72

[9] Mokhov O.I. 1995 Symplectic and poisson geometry on loop spaces of manifolds and nonlinear equations,Translations of the American Mathematical Society-Series **2 170** 121-152 [arXiv:hep-th/9503076]

[10] Jan Cieslinski The Darboux-Bianchi-Backlund transformation and soliton surfaces arXiv:1303.5472

[11] A. Sym 1982 Soliton Surfaces Lett. Nuovo Cimento **33** 394-400