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Soliton surfaces associated with the (1+1)-dimensional Yajima-Oikawa equation

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Abstract. Soliton surfaces associated with integrable systems play a significant role in physics and mathematics. In this paper, we investigate the relationship between integrable equations and differential geometry of surface by the example of the Yajima-Oikawa equation. The integrability of nonlinear equations is understood as the existence their Lax representations. Using the connection between classical geometry and soliton theory, we have found the soliton surface related with the Yajima-Oikawa equation. The surface area, curvature, the first and second fundamental forms are found.

1. Introduction

Nonlinear wave interactions lead to many interesting physical phenomena in various fields of science, including nonlinear optics, plasma physics, theory condensed matter and biophysics [1]-[6]. Among the systems of coupled wave dynamics is long- short-wave resonance interaction is a fascinating physical the process in which the resonant interaction between the weakly dispersing long-wave and short-wave packet, when the phase velocity of the first exactly coincides with the group velocity of the second. A theoretical study of this long-short-wave resonance interaction was first performed by Zakharov on a plasma of Langmuir waves. In the case of the propagation of long waves in one direction, the General Zakharov system was reduced to the Yajima-Oikawa (YO) system [7]-[8]. The system consists of two differential equations [9]:

$$iE_t + \frac{1}{2}E_{xx} - mE = 0, \tag{1}$$

$$m_t + m_x + |E|^2_x = 0, \tag{2}$$

where E - short-wave, m - long-wave components of the resonance interaction, indices x and t denote partial derivatives of the arguments x and t , i - imaginary unit. Lax representation of the system of YO equations (1)-(2) has next form:

$$\Psi_x = U\Psi, \tag{3}$$

$$\Psi_t = V\Psi, \tag{4}$$

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where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{5}$$

and

$$U = A_0 + 2i\lambda A_1 + (2\lambda)^{-1} A_{-1}, \tag{6}$$
$$V = -U + 2i\lambda^2 A_1^2 + \lambda B_1 + B_0 + i(4\lambda)^{-1} B_{-1}. \tag{7}$$

Here

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -\phi^* & 0 \\ 0 & 0 & 0 \\ 0 & \phi^* & 0 \end{pmatrix},$$
$$A_{-1} = \begin{pmatrix} -mi & 0 & -mi \\ \phi & 0 & \phi \\ mi & 0 & mi \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -\phi^* & 0 \\ 0 & 0 & 0 \\ 0 & \phi^* & 0 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 0 & \frac{i}{2}\phi_x^* + \phi^* & 0 \\ \frac{1}{2}\phi & 0 & -\frac{1}{2}\phi \\ 0 & -\frac{i}{2}\phi_x^* - \phi^* & 0 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} |\phi|^2 & 0 & |\phi|^2 \\ \phi_x & 0 & \phi_x \\ -|\phi|^2 & 0 & -|\phi|^2 \end{pmatrix}, \tag{8}$$

where

$$\phi = E \exp\left\{ i \left(\frac{t}{2} - x \right) \right\}. \tag{9}$$

The symbol * denotes complex conjugation. λ - complex parameter of eigenvalues (constant).

2. The first fundamental form of the surface

If the surface is introduced curvilinear coordinates, then we say that the surface is parameterized. In this case, any pair of x and t parameter values corresponds to a certain surface point. Consider a parameterized surface

$$\mathbf{r} = \mathbf{r}(x, t). \tag{10}$$

The dependence of the radius vector of a parameterized surface point on the curvilinear coordinates of this point is called the parametric equation of the surface. The full differential $d\mathbf{r}$ of the radius-vector $\mathbf{r}$ of the current surface is represented as a vector differential form

$$d\mathbf{r} = \mathbf{r}_x dx + \mathbf{r}_t dt. \tag{11}$$

The scalar square of this form is a scalar quadratic differential form [10]

$$d\mathbf{r}^2 = d\mathbf{r} d\mathbf{r} = \mathbf{r}_x^2 dx^2 + 2\mathbf{r}_x \mathbf{r}_t dx dt + \mathbf{r}_t^2 dt^2. \tag{12}$$

The scalar square of the total differential $d\mathbf{r}$ of the radius vector of the current surface point is called the first I fundamental form of the surface:

$$I = d\mathbf{r}^2. \tag{13}$$

The basic idea of the soliton surfaces approach is to relate to a given integrable system of nonlinear partial differential equations a class of surfaces immersed in the Lie algebra of a corresponding linear problem using the so-called Sym-Tafel formula [11]

$$r = \Phi^{-1}\Phi_\lambda. \tag{14}$$

Enter the notation

$$E = \mathbf{r}_x^2, \quad F = \mathbf{r}_x \mathbf{r}_t, \quad G = \mathbf{r}_t^2. \tag{15}$$

where

$$r_x = \Phi^{-1} U_\lambda \Phi, \tag{16}$$

$$r_t = \Phi^{-1} V_\lambda \Phi, \tag{17}$$

$$r_x r_t = \Phi^{-1} U_\lambda V_\lambda \Phi \tag{18}$$

and determine the coefficients E, F, G. By using equations (6)-(7) the values are found:

$$r_x^2 = \Phi^{-1} \left(2iA_1 - \frac{1}{2\lambda^2} A_{-1} \right)^2 \Phi, \tag{19}$$

$$r_t^2 = \Phi^{-1} \left(-2iA_1 + \frac{1}{2\lambda^2} A_{-1} + 4i\lambda A_1^2 + B_1 - \frac{1}{4\lambda^2} iB_{-1} \right)^2 \Phi, \tag{20}$$

$$r_x r_t = \Phi^{-1} \left(2iA_1 - \frac{1}{2\lambda^2} A_{-1} \right) \left(-2iA_1 + \frac{1}{2\lambda^2} A_{-1} + 4i\lambda A_1^2 + B_1 - \frac{1}{4\lambda^2} iB_{-1} \right) \Phi, \tag{21}$$

substituting (19)-(21) into equation (15) we obtain:

$$E = \frac{1}{2} \operatorname{tr} \left(r_x^2 \right) = \frac{1}{2} \operatorname{tr} \left(U_\lambda^2 \right), \tag{22}$$

$$G = \frac{1}{2} \operatorname{tr} \left(r_t^2 \right) = \frac{1}{2} \operatorname{tr} \left(V_\lambda^2 \right), \tag{23}$$

$$F = \frac{1}{2} \operatorname{tr} \left(r_x r_t \right) = \frac{1}{2} \operatorname{tr} \left(U_\lambda V_\lambda \right). \tag{24}$$

Let's rewrite equations (12)

$$I = \frac{1}{2} \left(\operatorname{tr} \left(U_\lambda^2 \right) dx^2 + 2 \operatorname{tr} \left(U_\lambda V_\lambda \right) dxdt + \operatorname{tr} \left(V_\lambda^2 \right) dt^2 \right). \tag{25}$$

Find alternately the terms of the equation (25)

$$\operatorname{tr} \left(U_\lambda^2 \right) = -8 - \frac{4m}{\lambda^2}, \tag{26}$$

$$\operatorname{tr} \left(V_\lambda^2 \right) = -8 - \frac{4m}{\lambda^2} - \frac{2 |\phi|^2}{\lambda^2} - 32\lambda^2, \tag{27}$$

$$\operatorname{tr} \left(U_\lambda V_\lambda \right) = 8 + \frac{4m}{\lambda^2} + \frac{|\phi|^2}{\lambda^2}. \tag{28}$$

Substituting equations (26)-(28) into equation (12), we obtain

$$I = \left(-4 - \frac{2m}{\lambda^2} \right) dx^2 + \left(4 + \frac{2m}{\lambda^2} + \frac{|\phi|^2}{2\lambda^2} \right) dxdt + \left(-4 - \frac{2m}{\lambda^2} - \frac{|\phi|^2}{\lambda^2} - 16\lambda^2 \right) dt^2. \tag{29}$$

Expanded view of the first fundamental form

$$I = Edx^2 + 2Fdxdt + Gdt^2. \tag{30}$$

2.1. Surface area

Consider some region S on the surface $\mathbf{r} = \mathbf{r}(x,t)$. The area S is determined by the formula [12]:

$$S = \int\int |\mathbf{r}_x \times \mathbf{r}_t|\, dxdt. \tag{31}$$

We formulate the modulus of the vector product via scalar products

$$|\mathbf{r}_x \times \mathbf{r}_t| = \sqrt{\mathbf{r}_x^2\mathbf{r}_t^2 - (\mathbf{r}_x\mathbf{r}_t)^2}. \tag{32}$$

Having expressions for the coefficients E, F, G of the first fundamental form, we obtain

$$|\mathbf{r}_x \times \mathbf{r}_t| = \sqrt{EG - F^2}. \tag{33}$$

Respectively,

$$S = \int\int \sqrt{EG - F^2} dxdt, \tag{34}$$

$$S = \int\int \sqrt{32m + 64\lambda^2 - \frac{2m\,|\phi|^2}{\lambda^4} - \frac{|\phi|^4}{\lambda^4} + \frac{4}{\lambda^2}} dxdt. \tag{35}$$

3. The second fundamental form of the surface

The second II fundamental form of the surface is called the scalar product of the full differential of the second order $d^2\mathbf{r}$ of the radius vector $\mathbf{r}$ of the current surface point on the normal n at this point [13] :

$$d^2\mathbf{r} = -d\mathbf{r}d\mathbf{n} = -\left[(\mathbf{r}_{xx}\mathbf{n})\, dx^2 + 2\,(\mathbf{r}_{xt}\mathbf{n})\, dxdt + (\mathbf{r}_{tt}\mathbf{n})\, dt^2\right] \tag{36}$$

or

$$II = -\left[(\mathbf{r}_{xx}\mathbf{n})\, dx^2 + 2\,(\mathbf{r}_{xt}\mathbf{n})\, dxdt + (\mathbf{r}_{tt}\mathbf{n})\, dt^2\right]. \tag{37}$$

Enter the notation

$$e = -\mathbf{r}_{xx}\mathbf{n}, \quad f = -\mathbf{r}_{xt}\mathbf{n}, \quad g = -\mathbf{r}_{tt}\mathbf{n}. \tag{38}$$

Determine the coefficients e, f, g

$$e = -\frac{1}{2}\,\mathrm{tr}(r_{xx}\cdot n), \tag{39}$$

$$g = -\frac{1}{2}\,\mathrm{tr}(r_{tt}\cdot n), \tag{40}$$

$$f = -\frac{1}{2}\,\mathrm{tr}(r_{xt}\cdot n). \tag{41}$$

Similarly, by using the Sym-Tafel equation we find

$$r_{xx} = \Phi^{-1}U_{\lambda x}\Phi + \Phi^{-1}\left[U_\lambda, U\right]\Phi, \tag{42}$$

$$r_{xt} = \Phi^{-1}U_{\lambda t}\Phi + \Phi^{-1}\left[U_\lambda, V\right]\Phi, \tag{43}$$

$$r_{tt} = \Phi^{-1}V_{\lambda t}\Phi + \Phi^{-1}\left[V_\lambda, V\right]\Phi. \tag{44}$$

Next, we define normal

$$n = \frac{\Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi}{\sqrt{\frac{1}{2}\,\mathrm{tr}\left(\left[U_\lambda, V_\lambda\right]^2\right)}}. \tag{45}$$

The final form of the coefficients of the second fundamental form has next form:

$$e = -\frac{1}{2}\frac{\operatorname{tr}\left(\left(U_{\lambda x}+[U_\lambda,U]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}\left([U_\lambda,V_\lambda]^2\right)}}, \tag{46}$$

$$f = -\frac{1}{2}\frac{\operatorname{tr}\left(\left(U_{\lambda t}+[U_\lambda,V]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}\left([U_\lambda,V_\lambda]^2\right)}}, \tag{47}$$

$$g = -\frac{1}{2}\frac{\operatorname{tr}\left(\left(V_{\lambda t}+[V_\lambda,V]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}\left([U_\lambda,V_\lambda]^2\right)}}. \tag{48}$$

We find the commutators of equation (46)-(48) from equation (6) and (7)

$$[U_\lambda,V_\lambda] = 2i\left(A_1B_1-B_1A_1\right)+\frac{1}{2\lambda^2}\left(A_1B_{-1}-A_{-1}B_1+B_1A_{-1}-B_{-1}A_1\right)+ \\ +\frac{i}{8\lambda^4}\left(A_{-1}B_{-1}-B_{-1}A_{-1}\right), \tag{49}$$

$$[U_\lambda,U] = 2iA_1A_0-\frac{1}{2\lambda^2}A_1A_0+\frac{i}{\lambda}\left(A_1A_{-1}-A_{-1}A_1\right)+\frac{1}{4\lambda^3}\left(A_{-1}A_1-A_1A_{-1}\right), \tag{50}$$

$$[V_\lambda,V] = 2i\left(A_1A_0-A_1B_0+B_0A_1+\lambda^2A_1^2B_1\right)+4i\lambda\left(A_1^2B_0-A_1^2A_0-B_0A_1^2\right)+ \\ +\frac{2i}{\lambda}\left(A_1A_{-1}-A_{-1}A_1\right)+\frac{1}{\lambda}\left(A_1B_{-1}-B_1A_{-1}-B_{-1}A_1\right)+\frac{3}{2}\left(B_{-1}A_1^2-A_1^2B_{-1}\right)+ \\ +\frac{1}{2\lambda^2}\left(A_{-1}B_0+A_0A_{-1}-B_0A_{-1}\right)+\frac{i}{4\lambda^2}\left(B_0B_{-1}-B_{-1}B_0-A_0B_{-1}\right)+B_1B_0- \\ -B_0B_1+\frac{i}{2\lambda}B_1B_{-1}, \tag{51}$$

$$[U_\lambda,V] = 2i\left(\lambda A_1B_1-\lambda B_1A_1-A_1A_0+A_1B_0-B_0A_1\right)+i\left(A_1^2A_{-1}-A_{-1}A_1^2\right)+ \\ +\frac{1}{2\lambda^2}\left(B_0A_{-1}-A_{-1}B_0-A_0A_1\right)+\frac{1}{2\lambda}\left(B_1A_{-1}+B_{-1}A_1-A_1B_1\right)+\frac{i}{8\lambda^3}B_{-1}A_{-1}+ \\ +\frac{2i}{\lambda}\left(A_{-1}A_1-A_1A_{-1}\right) \tag{52}$$

also

$$U_{\lambda x} = -\frac{1}{2\lambda^2}\begin{pmatrix} -im_x & 0 & -im_x \\ \phi_x & 0 & \phi_x \\ im_x & 0 & im_x \end{pmatrix}, \tag{53}$$

$$U_{\lambda t} = -\frac{1}{2\lambda^2}\begin{pmatrix} -im_t & 0 & -im_t \\ \phi_t & 0 & \phi_t \\ im_t & 0 & im_t \end{pmatrix}, \tag{54}$$

$$V_{\lambda t} = \begin{pmatrix} -\frac{im_t}{2\lambda^2}-\frac{i|\phi|_t^2}{4\lambda} & -\phi_t^* & -\frac{im_t}{2\lambda^2}-\frac{i|\phi|_t^2}{4\lambda} \\ \frac{\phi_t}{2\lambda^2}-\frac{i\phi_{xt}}{4\lambda} & 0 & \frac{\phi_t}{2\lambda^2}-\frac{i\phi_{xt}}{4\lambda} \\ \frac{im_t}{2\lambda^2}+\frac{i|\phi|_t^2}{4\lambda} & \phi_t^* & \frac{im_t}{2\lambda^2}+\frac{i|\phi|_t^2}{4\lambda} \end{pmatrix}. \tag{55}$$

Now, the equations (50)-(55) we substitute in (46)-(48)

$$e = -\frac{4m}{\lambda} - \frac{im}{\lambda^3} - \frac{2i\phi_x}{\phi}, \tag{56}$$

$$g = -4\lambda^2 - 2|\phi|^2 - \frac{4|\phi|^2}{\lambda} - \frac{8m}{\lambda} - \frac{\phi_t^*\phi_x}{|\phi|^2} - \frac{2\lambda|\phi_x|^2}{|\phi|^2} + \frac{8i\lambda\phi_x}{\phi} + \frac{2i\phi_t}{\phi} + \frac{\lambda\phi_{xt}}{\phi} + \frac{4i\lambda^2\phi_x}{\phi}, \tag{57}$$

$$f = 10m + 8\lambda^2 + \frac{9|\phi|^2}{4\lambda} + \frac{|\phi|^2 - im}{2\lambda^2} - \frac{2\phi_t}{\phi}, \tag{58}$$

and can get the second fundamental form in following form

$$II = edx^2 + 2fdxdt + gdt^2. \tag{59}$$

4. Total and mean curvatures of the surface

The Gaussian or total curvature as well as the mean curvature can be found directly by the first and second fundamental forms. The Gaussian curvature of the surface is equal to the ratio of the determinants of the second and first fundamental forms [14]:

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}, \tag{60}$$

mean curvature of the surface

$$H = \frac{1}{2}\frac{Eg + Ge - 2Ff}{EG - F^2}. \tag{61}$$

5. Conclusion

In this paper, we consider the YO equation, the integrability of which is realized by the assumption of the Lax representation for it. We investigate the construction of a surface for the 1+1-dimensional YO equation. The first and second fundamental forms of the surface with the corresponding coefficients are found. The Sym-Tafel formula and the theory of differential geometry of the surface are applied. The area, Gaussian and mean curvatures of the surface are obtained.

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6. References

[1] Bekova G, Shaikhova G and Yesmakhanova K 2018 *AIP Conf. Proc.* **1997** 020039 (1–6)
[2] Shaikhova G, Ozat N, Yesmakhanova K and Bekova G 2018 *J. Phys.: Conf. Series* **965** 012035 (1–10)
[3] Shaikhova G, Yesmakhanova K, Bekova G and Ybyraiymova S 2017 *J. Phys.: Conf. Series* **936** 012098 (1–6)
[4] Yesmakhanova K, Shaikhova G, Bekova G, and Myrzakulov R 2017 *AIP Conf. Proc.* **1880** 060022 (1–7)
[5] Bekova G, Shaikhova G, Yesmakhanova K and Myrzakulov R 2017 *J. Phys.: Conf. Series* **804** 012004 (1–8)
[6] Shaikhova G 2018 *Bul. Univ. Krg. - Math.* **92 (4)** (94–98)
[7] Zakharov V E 1972 *Soviet physics jetp 35* **5** 908
[8] Chen J, Chen Y, Feng B F, Maruno K, Ohta Y 2018 *The Physical Society of Japan* **9** 87 094007
[9] Makhankov V G, Myrzakulov R 1984 *JINR* (Dubna) p 6
[10] Rogers C, Schief W K 2002 *Backlund and Darboux transformations: geometry and modern applications in soliton theory* (Cambridge University Press) p 18
[11] Cieslinski J, Wojcik D 1998 The Darboux-Bianchi-Backlund transformation and soliton surfaces *Nonlinearity and Geometry, Publisher: Polish Scientific Publisher PWN, Warsaw* 81-107

[12] Nutbourne A W, Martin R R 1988 *Differential geometry applied to the design of curves and surfaces* (Ellis Horwood, Chichester) p 198-245
[13] Ignatiev Yu G 2013 *Differential geometry of curves and surfaces in Euclidean space* (Kazan University) p 204
[14] Cieslinski J, Goldstein P, Sym A 1995 *Phys.Lett. A. 205* 154