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Soliton surfaces for complex modified Korteweg−de Vries equation

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Abstract.   In mathematics and physics, one of the main tasks is to relate differential geometry and non-linear differential equations, which means that the study of particular cases of subvarieties, curves, and surfaces are of great importance. Soliton surfaces associated with the integrable system play an essential role in many problems with the physical application. In this paper, we study the complex modified Korteweg−de Vries (cmKdV) equation. It is well known that the cmKdV equation is a very important integrable equation. We present the relationship between an integrable system and soliton surfaces and namely Lax representation of the cmKdV equation was used to obtain the first and second fundamental forms, surface area and curvature.

1. Introduction

The term soliton - got its name from the word solitary wave, which is a localized wave, that arises from the balance between nonlinear and dispersion effects. Despite initial research, the solitary wave concept could not gain much recognition for many years. Korteweg− de Vries (1895) developed a mathematical model for the shallow water problem and showed the possibility of a solitary wave. Later, the study of solitary waves began in the mid-1960s, when the Zabusky and Kruskal discovered stable behavior similar to a particle of solitary waves [1]-[4]. Earlier considered the real modified Korteweg−de Vries[5]. We considered here the complex modified Korteweg−de Vries equation. Soliton is a structural stable solitary wave propagating in a nonlinear medium. Solitons behave like particles when interacting with each other, they are not destroyed but continue to move to maintain their structure unchanged. Finding solutions to complex modified Korteweg-de Vries (cmKdV) is of great importance because solutions help to understand well the complex mechanisms of physical phenomena and dynamic processes. Partial nonlinear equations are the cmKdV equation [6]-[11]. We begin with coupled cmKdV equations of the form of [12]

$$q_t + q_{xxx} - 6qrq_x = 0, \tag{1}$$

$$r_t + r_{xxx} - 6rqr_x = 0. \tag{2}$$

We introduce $q = r^*$ equation (1)-(2) has next form

$$q_t + q_{xxx} + 6\left|q\right|^2 q_x = 0. \tag{3}$$

Here q (x, t) - the deviation from the equilibrium position of the water surface waveform depends on the x coordinate and time t. The indices of the characteristic q function mean the corresponding derivatives concerning t and x. This equation is a partial differential equation. The characterization under study in this case, q depends on the coordinate x and time t. To solve an equation of this type means to find the dependence of q on x and t, after substituting it into the equation we come to identify the modified Korteweg-de Vries equations that will be introduced in this chapter. Lax representation of the cmKdV equations has next form [13]-[14]

$$\Psi_x \ = \ M\Psi, \tag{4}$$
$$\Psi_t \ = \ N\Psi, \tag{5}$$

where ψ-vector

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{6}$$

and M,N are matrices

$$M = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, \tag{7}$$

$$N = V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0. \tag{8}$$

Here

$$V_3 = \begin{pmatrix} -4i & 0 \\ 0 & 4i \end{pmatrix}, \tag{9}$$

$$V_2 = \begin{pmatrix} 0 & 4q \\ 4r & 0 \end{pmatrix}, \tag{10}$$

$$V_1 = \begin{pmatrix} -2iqr & 2iq_x \\ -2ir_x & 2iqr \end{pmatrix}, \tag{11}$$

$$V_0 = \begin{pmatrix} -qr_x + q_xr & -q_{xx} + 2q^2r \\ -r_{xx} + 2qr^2 & qr_x - q_xr \end{pmatrix}, \tag{12}$$

$$\tag{13}$$

where, λ - complex parameter of eigenvalues constant.

2. Fundamental form.

In this chapter, we used the first and second fundamental form for finding soliton surfaces and used the Sym Tafel formula. The Sym Tafel formula gives the connection between the theory of solitons and classical geometry Finding soliton surfaces is important when solving integrable geometry. Geometrical objects associated with soliton surfaces can be associated with the solutions of some nonlinear models [15]-[19].

Using Sym-Tafel formula

$$r = \Phi^{-1}\Phi_\lambda, \tag{14}$$

we find the next;

$$r_x = \Phi^{-1}M_\lambda\Phi, \tag{15}$$
$$r_t = \Phi^{-1}N_\lambda\Phi. \tag{16}$$

where,

$$M_\lambda = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{17}$$

$$N_\lambda = \begin{pmatrix} -12i\lambda^2 - 2iqr & 8q\lambda + 2iq_x \\ 8r\lambda - 2ir_x & 12i\lambda^2 + 2iqr \end{pmatrix}. \tag{18}$$

$$\tag{19}$$

2.1. The first fundamental form of the surface

Consider a parameterized surface

$$r = r(x, t). \tag{20}$$

Regardless of whether the parameters (x,t) are independent arguments or any functions of other independent arguments, the total dierential dr of the radius vector r of the current surface point is represented as a (vector) invariant linear dierential form

$$dr = r_x dx + r_t dt. \tag{21}$$

Which is a scalar quadratic differential form, has the same invariance property

$$dr^2 = drdr = r_x^2 dx^2 + 2r_x r_t dxdt + r_t^2 dt^2. \tag{22}$$

In expanded form, we can write in the following form

$$\phi_1 = Edx_2 + 2Fdxdt + Gdt^2, \tag{23}$$

where E,F,G - the first quadratic form are determined from the previously obtained expression

$$E = r_x^2; F = r_x r_t; G = r_t^2, \tag{24}$$

where

$$r_x = \Phi^{-1}M_\lambda\Phi, \tag{25}$$
$$r_t = \Phi^{-1}N_\lambda\Phi, \tag{26}$$
$$r_x r_t = \Phi^{-1}M_\lambda N_\lambda\Phi. \tag{27}$$

Consider two-dimentional surface

$$\vec{r}_x^{\,2} = -\frac{1}{2}tr(r_x^2), \tag{28}$$

$$\vec{r}_t^{\,2} = -\frac{1}{2}tr(r_t^2), \tag{29}$$

$$\vec{r}_x\vec{r}_t = -\frac{1}{2}tr(r_xr_t), \tag{30}$$

we find

$$r_x^2 = \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right), \tag{31}$$

$$tr(\vec{r}_x^{\,2}) = -2, \tag{32}$$

$$r_t^2 = \left(\begin{array}{c} -144\lambda^4 - 48\lambda^2 qr - 4q^2r^2 + 64\lambda^2 qr - 16q\lambda i r_x + 16q_x ir\lambda + 4q_x r_x \\ -96iq\lambda^3 + 24\lambda^2 q_x - 16iq^2 r\lambda + 4rqq_x + 96qi\lambda^3 + 16iq^2\lambda r - 24\lambda^2 q_x - 4rqq_x \end{array} \right.$$

$$\left. \begin{array}{c} -96\lambda^3 ir - 16ir^2 q - 24\lambda^2 r_x - 4qr_x r + 96ri\lambda^3 - 24\lambda^2 r_x + 16iqr^2\lambda - 4qrr_x \\ 64\lambda^2 rq + 16iq_x r\lambda - 16q\lambda ir_x + 4q_x r_x - 144\lambda^4 - 48\lambda^2 qr - 4q^2r^2 \end{array} \right), \tag{33}$$

$$tr(\vec{r}_t^{\,2}) = -8q^2r^2 - 32qr\lambda^2 + 64rq\lambda^2 - 288\lambda^4 + 32r\lambda iq_x - 32q\lambda ir_x + 8q_x r_x, \tag{34}$$

$$r_x r_t = \left(\begin{array}{cc} -12\lambda^2 - 2qr & 8iq\lambda - 2q_x \\ 8ir\lambda + 2r_x & -12\lambda^2 - 2qr \end{array} \right), \tag{35}$$

$$tr(\vec{r}_t^{\,2}) = -4qr - 24\lambda^2. \tag{36}$$

We rewrite first foundation form equations as

$$I = \frac{1}{2}\left(\mathrm{tr}(M_\lambda^2)dx^2 + 2\,\mathrm{tr}(M_\lambda N_\lambda)dxdt + \mathrm{tr}(N_\lambda^2)dt^2 \right), \tag{37}$$

where

$$\mathrm{tr}\left(M_\lambda^2 \right) = -2, \tag{38}$$

$$\mathrm{tr}\left(N_\lambda^2 \right) = -8q^2r^2 - 32qr\lambda^2 + 64rq\lambda^2 - 288\lambda^4 + 32r\lambda iq_x - 32q\lambda ir_x + 8q_x r_x, \tag{39}$$

$$\mathrm{tr}\left(M_\lambda N_\lambda \right) = -4qr - 24\lambda^2. \tag{40}$$

Substituting equations (38)-(40) into equation (37), we obtain

$$I = dx^2 - [-24i\lambda^2 - 4qr]dxdt + [-8q^2r^2 - 32qr\lambda^2 + 64rq\lambda^2 - \tag{41}$$
$$-288\lambda^4 + 32r\lambda iq_x - 32q\lambda ir_x + 8q_x r_x]dt.$$

Expanded view of the first fundamental form has next view

$$I = Edx^2 + 2Fdxdt + Gdt^2. \tag{42}$$

2.2. The second fundamental form of the surface

The second quadratic form (or second fundamental form) of the surface is a quadratic form on the tangent bundle of the surface, which, unlike the first quadratic form, defines the external geometry of the surface in the vicinity of this point

$$II = -\left[\left(\vec{r}_{xx}\vec{n}\right)dx^2 + (2\vec{r}_{xt}\vec{n})dxdt + (\vec{r}_{tt}\vec{n})dt^2\right]. \tag{43}$$

where

$$\vec{r}_{xx}\vec{n} = -\frac{1}{2}tr(r_{xx}n), \tag{44}$$

$$\vec{r}_{tt}\vec{n} = -\frac{1}{2}tr(r_{tt}n), \tag{45}$$

$$\vec{r}_{xt}\vec{n} = -\frac{1}{2}tr(r_{xt}n). \tag{46}$$

$$\tag{47}$$

We introduce the notation

$$e = \vec{r}_{xx}\vec{n}, \quad f = \vec{r}_{tt}\vec{n}, \quad g = \vec{r}_{xt}\vec{n}. \tag{48}$$

Next, using the equations of Sym Tafel we find

$$r_{xx} = \Phi^{-1}M_{\lambda x}\Phi + \Phi^{-1}\left[M_\lambda, M\right]\Phi, \tag{49}$$

$$r_{xt} = \Phi^{-1}M_{\lambda t}\Phi + \Phi^{-1}\left[M_\lambda, N\right]\Phi, \tag{50}$$

$$r_{tt} = \Phi^{-1}N_{\lambda t}\Phi + \Phi^{-1}\left[N_\lambda, N\right]\Phi. \tag{51}$$

we define normal

$$n = \frac{\Phi^{-1}\left[M_\lambda, N_\lambda\right]\Phi}{\sqrt{\frac{1}{2}\,\mathrm{tr}\left(\left[M_\lambda, N_\lambda\right]^2\right)}}. \tag{52}$$

The second fundamental form has next form

$$e = -\frac{1}{2}\frac{\mathrm{tr}\left(\left(M_{\lambda x} + \left[M_\lambda, M\right]\right)\left[M_\lambda, N_\lambda\right]\right)}{\sqrt{\frac{1}{2}\,\mathrm{tr}\left(\left[M_\lambda, N_\lambda\right]^2\right)}}, \tag{53}$$

$$f = -\frac{1}{2}\frac{\mathrm{tr}\left(\left(M_{\lambda t} + \left[M_\lambda, N\right]\right)\left[M_\lambda, N_\lambda\right]\right)}{\sqrt{\frac{1}{2}\,\mathrm{tr}\left(\left[M_\lambda, N_\lambda\right]^2\right)}}, \tag{54}$$

$$g = -\frac{1}{2}\frac{\mathrm{tr}\left(\left(N_{\lambda t} + \left[N_\lambda, N\right]\right)\left[M_\lambda, N_\lambda\right]\right)}{\sqrt{\frac{1}{2}\,\mathrm{tr}\left(\left[M_\lambda, N_\lambda\right]^2\right)}}. \tag{55}$$

Further from the equation (53)-(55) we find the commutators;

$$[M_\lambda, M] = \begin{pmatrix} 0 & -2iq \\ 2ir & 0 \end{pmatrix}, \tag{56}$$

$$[M_\lambda, N_\lambda] = \begin{pmatrix} 0 & -2i(8q\lambda + 2q_x) \\ 2i(8r\lambda - 2r_x) & 0 \end{pmatrix}, \tag{57}$$

$$[M_\lambda, N] = \begin{pmatrix} 0 & -8\lambda^2 i + 4q_x\lambda + 2q_{xx}i - 4q^2ri \\ 8\lambda^2 i + 4r_x\lambda - 2r_{xx}i + 4qr^2i & 0 \end{pmatrix}, \tag{58}$$

$$[N_\lambda, N] = \left(\begin{matrix} -4q^2r^2\lambda + 2iq^2rr_x - 2iqq_xr - 32\lambda^3 qr + 12qr_xi\lambda^2 - 12q_xri\lambda^2 + 48i\lambda^5 \\ 16qr^3\lambda - 8r_{xx}r\lambda + 32r\lambda^3 - 4qr^3ir_x + 2r_{xx}ir_x - 8r\lambda^2ir_x - 16r\lambda^2ir_x - 4r_x^2\lambda \end{matrix} \right.$$

$$\left. \begin{matrix} -8q_{xx}q\lambda + 32q\lambda^3 + 16q^3\lambda r - 2q_{xx}q_xi + 8q\lambda^2 iq_x + 4q^2riq_x + 16qq_x\lambda^2 i - 4qq_x\lambda \\ -4q^2r^2 + 2q^2rr_xi - 2q_xr^2iq - 32i\lambda^3 qr + 12qr_xi\lambda^2 - 12q_xri\lambda^2 + 48i\lambda^5 \end{matrix} \right), \tag{59}$$

also

$$M_{\lambda x} = 0, \tag{60}$$

$$M_{\lambda t} = 0, \tag{61}$$

$$N_{\lambda t} = \begin{pmatrix} -2i(rq_t + qr_t) & 8q\lambda q_t + 2iq_{xt} \\ 8r_t\lambda - 2ir_{xt} & 2i(q_tr + qr_t) \end{pmatrix}, \tag{62}$$

we substituting (56)-(62) and can get the second fundamental form in following form

$$II = edx^2 + 2fdxdt + gdt^2. \tag{63}$$

3. Surface area

In all area definitions, the first-class describes the surface class. It is easiest to determine the area of polyhedral surfaces: as the sum of the areas of their flat faces. Not too wide for most applications. Most often, the surface area is determined for the class of piecewise smooth surfaces with a piecewise smooth edge. This can be done using the following construction: The surface is divided into parts with piecewise smooth boundaries: for each part, a plane is selected and the part in question is orthogonally projected onto it; the area of the obtained flat projections is summarized. The surface area itself is defined as the exact upper bound of such sums. If a surface in Euclidean space is given parametrically function $r(x, t)$, where are the parameters x, t in area D on surface x, t so area S can be expressed as a double integral

$$S = \int\int \left| \vec{r_x} \times \vec{r_t} \right| dxdt. \tag{64}$$

where $\times$ - a vector product, r_x, r_t-private derivatives by x, t and

$$\left| \vec{r_x} \times \vec{r_t} \right| = \sqrt{\vec{r_x}^2\vec{r_t}^2 - (\vec{r_x}\vec{r_t})^2}, \tag{65}$$

$$\left| \vec{r_x} \times \vec{r_t} \right| = \sqrt{EG - F^2}. \tag{66}$$

We can write

$$S = \int\int \sqrt{EG - F^2}dxdt, \tag{67}$$

where:

$$E = \frac{1}{2}tr(r_x^2) = \frac{1}{2}tr(M_\lambda^2) = 1, \tag{68}$$

$$G = \frac{1}{2}tr(r_t^2) = \frac{1}{2}tr(N_\lambda^2) = -8q^2r^2 - 32qr\lambda^2 + 64rq\lambda^2 - 288\lambda^4 + 32r\lambda iq_x - 32q\lambda ir_x + 8q_xr_x, \tag{69}$$

$$F = \frac{1}{2}tr(r_{xt}) = \frac{1}{2}tr(M_\lambda N_\lambda) = 0, \tag{70}$$

$$S = \int\int \sqrt{-8q^2r^2 - 32qr\lambda^2 + 64rq\lambda^2 - 288\lambda^4 + 32r\lambda iq_x - 32q\lambda ir_x + 8q_xr_x}dt. \tag{71}$$

4. Total and mean curvatures of the surface

In studying the properties of regular surfaces, the concepts of average surface curvature and Gaussian curvature are widely used. The average curvature of the surface at a given point is the half-sum of its main curvatures

$$H = \frac{1}{2}(k_1 + k_2). \tag{72}$$

The Gaussian curvature of a surface is the product of its principal curvatures

$$K = k_1k_2, \tag{73}$$

using the properties of the roots of the quadratic equation, we obtain the following formulas for the average curvature H and the Gaussian curvature K:

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}, \tag{74}$$

$$H = \frac{1}{2}\frac{Eg + Ge - 2Ff}{EG - F^2}. \tag{75}$$

5. Conclusion

In this article, we examined the complex modified KdV equation. For integrability, we introduced the Lax pair and investigated a one-dimensional surface. The first and second fundamental forms were found in the formula of Sym Tafel. We found the surface area, Gaussian and average surface curvature.

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7. References

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