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Soliton surfaces induced by the Fokas-Lenells equation

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Abstract. In this paper, we study the application of the theory of solitons in differential geometry. The recently proposed soliton equation, which is Fokas-Lenells equation, has been investigated, and its two-dimensional soliton surface in the three-dimensional Euclidean space ($\mathbb{R}^2 \to \mathbb{R}^3$) has been constructed. Thus the connection between the Fokas-Lenells equation and the surface was established by using the Sym-Tafel formula. We find the first and the second quadratic forms, surface area, and Gaussian curvature. The obtained results have various applications in mathematical physics, the geometry of curves and the theory of surfaces.

1. Introduction

The study of properties as geometry [1]-[4], integrability [5], and exact [6]-[10] solutions of nonlinear equation have an impotant role in the application on of physics. The theory of surfaces in three-dimensional Euclidean space is widely used in various fields of science, in particular in mathematics, theoretical physics, etc. In this paper, we apply it to the theory of integrable system. To this end, the integrable Fokas-Lenells (FL) equation, which describes the propagation of ultrashort nonlinear light pulses in optical fibers and looks as follows, is investigated:

$$iq_{xt} - iq_{xx} + 2q_x - |q|^2 q_x + iq = 0, \quad (1)$$

$$ir_{xt} - ir_{xx} - 2r_x + |q|^2 r_x + ir = 0, \quad (2)$$

where $q(x,t)$ is the complex envelope of a field, the indices x, t denote partial derivatives with respect to the arguments x, t and i are an imaginary unit. In physical applications, two natural reductions are used ($r = \pm\bar{q}$), which are of great interest in physics, one of them is a of defocusing case, with $r = -\bar{q}$ [11]-[13]

$$FL_- : \quad iq_{xt} - iq_{xx} + 2q_x - |q|^2 q_x + iq = 0,$$

and second, the of focusing case, with $r = \bar{q}$

$$FL_+ : \quad iq_{xt} - iq_{xx} + 2q_x + |q|^2 q_x + iq = 0.$$

In this paper, we consider the first case, that is

$$iq_{xt} - iq_{xx} + 2q_x - |q|^2 q_x + iq = 0. \quad (3)$$

Since, the equation under consideration is integrable, it has the Lax pair (LP), which plays an important role in the theory of integrable systems. It allows you to apply the inverse scattering method to construct exact solutions and to study the asymptotics of problems with initial conditions. The LP for equation (3) has next form

$$\Phi_x(x,t,\lambda) = U(x,t,\lambda)\Phi(x,t,\lambda), \tag{4}$$
$$\Phi_t(x,t,\lambda) = V(x,t,\lambda)\Phi(x,t,\lambda), \tag{5}$$

where $\Phi = (\Phi_1, \Phi_2)^T$ is called 2×2 the matrix eigenfunction of the eigenvalue λ (or spectral parameter) and the matrix operators U and V are given in the following form:

$$U = -i\lambda^2\sigma_3 + \lambda Q, \tag{6}$$

$$V = -i\lambda^2\sigma_3 + \lambda Q + V_0 + \frac{1}{\lambda}V_{-1} - \frac{i}{4\lambda^2}\sigma_3. \tag{7}$$

Here

$$Q = \begin{pmatrix} 0 & q_x \\ \bar{q}_x & 0 \end{pmatrix}, \quad V_0 = i\sigma_3 - \frac{i|q|^2}{2}\sigma_3, \quad V_{-1} = \frac{i}{2}\begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. The first fundamental form of the surface

The first fundamental form (1FF) plays an important role in the theory of surfaces and serves primarily to measure infinitely small arcs on a surface. 1FF of a smooth surface P is called a scalar square the radius of the vector $d\mathbf{r} = \mathbf{r}_x dx + \mathbf{r}_t dt$, that is,

$$I = d\mathbf{r}^2 = (\mathbf{r}_x dx + \mathbf{r}_t dt)^2 = \mathbf{r}_x^2 dx^2 + 2\mathbf{r}_x\mathbf{r}_t dxdt + \mathbf{r}_t^2 dt^2. \tag{8}$$

The equation (8) at each point of the surface P represents the quadratic form of the differentials dx and dt.

For coefficients 1FF, the following notation is often used:

$$E = \mathbf{r}_x^2, \quad F = \mathbf{r}_x\mathbf{r}_t, \quad G = \mathbf{r}_t^2, \tag{9}$$

then the equation (8) is rewritten as

$$I = Edx^2 + 2Fdxdt + Gdt^2, \tag{10}$$

where $EG - F^2 > 0$.

Two-dimensional surfaces and integrable equations are relate by the Lax representation. This is done with the help of the so-called Sym-Tafel formula (STF), which has the form [12]

$$r = \Phi^{-1}\Phi_\lambda, \tag{11}$$

and defines a λ-family of surfaces parametrized by coordinates x and t. From equation (11) we can define 1FF. The surfaces are immersed in the linear space of the matrices from the equation (11) [14].

It is very convenient to apply the STF because of the very simple rule for calculating the derivatives with respect to r

$$r_x = (\Phi^{-1})_x\Phi_\lambda + \Phi^{-1}\Phi_{\lambda x} = -\Phi^{-1}\Phi_x\Phi^{-1}\Phi_\lambda + \Phi^{-1}(U\Phi)_\lambda =$$
$$= -\Phi^{-1}U\Phi_\lambda + \Phi^{-1}U_\lambda\Phi + \Phi^{-1}U\Phi_\lambda = \Phi^{-1}U_\lambda\Phi, \tag{12}$$

and similarly,

$$r_t = \Phi^{-1} V_\lambda \Phi. \tag{13}$$

Scalar product is standard, that is, proportional to the trace of the matrix. For example: $r_x \cdot r_t = c\,\mathrm{tr}(U_\lambda V_\lambda)$, where c is a constant. Then, in terms of matrix operators, the equation (8) is rewritten as follows:

$$I = \frac{1}{2}\left(\mathrm{tr}(U_\lambda^2)dx^2 + 2\,\mathrm{tr}(U_\lambda V_\lambda)dxdt + \mathrm{tr}(V_\lambda^2)dt^2\right). \tag{14}$$

Thus, it became obvious that the STF serves as a bridge for establishing a connection with the soliton equation and the surface.

Now, let's differentiate the matrix operators U and V by the spectral parameter λ, we take them in the square and find their trace, that is,

$$\mathrm{tr}(U_\lambda^2) = 2(|q_x|^2 - 4\lambda^2), \tag{15}$$

$$\mathrm{tr}(U_\lambda V_\lambda) = 2\left(|q_x|^2 - 4\lambda^2 + \frac{1}{\lambda^2}\right) + \frac{i}{2\lambda^2}(\bar{q}q_x - q\bar{q}_x), \tag{16}$$

$$\mathrm{tr}(V_\lambda^2) = 2\left(|q_x|^2 - 4\lambda^2 + \frac{i}{2\lambda^2}(\bar{q}q_x - q\bar{q}_x) + \frac{2}{\lambda^2} + \frac{1}{4\lambda^4}|q|^2 - \frac{1}{4\lambda^6}\right). \tag{17}$$

Substituting the equations (15)-(17) into the equation (14), we get

$$I = (|q_x|^2 - 4\lambda^2)dx^2 + \left(2(|q_x|^2 - 4\lambda^2 + \frac{1}{\lambda^2}) + \frac{i}{2\lambda^2}(\bar{q}q_x - q\bar{q}_x)\right)dxdt +$$
$$+ \left(|q_x|^2 - 4\lambda^2 + \frac{i}{2\lambda^2}(\bar{q}q_x - q\bar{q}_x) + \frac{2}{\lambda^2} + \frac{1}{4\lambda^4}|q|^2 - \frac{1}{4\lambda^6}\right)dt^2, \tag{18}$$

or as equation (10)

$$I = Edx^2 + 2Fdxdt + Gdt^2,$$

where

$$E = |q_x|^2 - 4\lambda^2, \tag{19}$$

$$F = E + \frac{1}{\lambda^2} + \frac{i}{4\lambda^2}(\bar{q}q_x - q\bar{q}_x), \tag{20}$$

$$G = E + \frac{i}{2\lambda^2}(\bar{q}q_x - q\bar{q}_x) + \frac{2}{\lambda^2} + \frac{1}{4\lambda^4}|q|^2 - \frac{1}{4\lambda^6}. \tag{21}$$

Thus, we found 1FF surfaces of the FL equation.

3. The second fundamental form of the surface

The second fundamental form (2FF) describes the surface in the second approximation. It shows how the surface deviates from the tangent plane and completely determines the curvature of the surface. 2FF regular surface f is the scalar product of vectors $d^2\mathbf{r}$ and $\mathbf{n}$:

$$II = (d^2\mathbf{r}\mathbf{n})^2, \tag{22}$$

where the unit normal vector is defined as

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_t}{|\mathbf{r}_x \wedge \mathbf{r}_t|} = \frac{\mathbf{r}_x \wedge \mathbf{r}_t}{\sqrt{EG - F^2}}$$

and the second differential of the vector function $\mathbf{r}(x,t)$

$$d^2\mathbf{r} = \mathbf{r}_{xx}dx^2 + 2\mathbf{r}_{xt}dxdt + \mathbf{r}_{tt}dt^2. \tag{23}$$

Then, the equation (22) will has written as

$$II = (\mathbf{r}_{xx}\cdot\mathbf{n})\,dx^2 + 2(\mathbf{r}_{xt}\cdot\mathbf{n})\,dxdt + (\mathbf{r}_{tt}\cdot\mathbf{n})\,dt^2. \tag{24}$$

For coefficients 2FF, the following notation is used:

$$L = \mathbf{r}_{xx}\cdot\mathbf{n}, \quad M = \mathbf{r}_{xt}\cdot\mathbf{n}, \quad N = \mathbf{r}_{tt}\cdot\mathbf{n}.$$

This allows you to rewrite the equation (24) as

$$II = Ldx^2 + 2Mdxdt + Ndt^2, \tag{25}$$

where the coefficients L, M and N are calculated as

$$L = \frac{1}{2}\operatorname{tr}(r_{xx}n), \quad M = \frac{1}{2}\operatorname{tr}(r_{xt}n), \quad N = \frac{1}{2}\operatorname{tr}(r_{tt}n), \tag{26}$$

where

$$n = \pm\frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{27}$$

and the second derivatives of the equation (12)-(13) can be easily calculated

$$r_{xx} = (\Phi^{-1}U_\lambda\Phi)_x = -\Phi^{-1}\Phi_x\Phi^{-1}U_\lambda\Phi + \Phi^{-1}U_{\lambda x}\Phi + \Phi^{-1}U_\lambda\Phi_x =$$
$$= -\Phi^{-1}UU_\lambda\Phi + \Phi^{-1}U_{\lambda x}\Phi + \Phi^{-1}U_\lambda U\Phi = \Phi^{-1}(-UU_\lambda + U_{\lambda x} + U_\lambda U)\Phi = \Phi^{-1}(U_{\lambda x} + [U_\lambda, U])\Phi, \tag{28}$$

and, by same way we can find,

$$r_{xt} = \Phi^{-1}(U_{\lambda t} + [U_\lambda, V])\Phi, \tag{29}$$
$$r_{tt} = \Phi^{-1}(V_{\lambda t} + [V_\lambda, V])\Phi. \tag{30}$$

Thus, substituting the equations (28)-(30) into the equation (26) we get the final formula for calculating the coefficients 2FF

$$L = \frac{1}{2}\frac{\operatorname{tr}\left((U_{\lambda x} + [U_\lambda, U])\,[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{31}$$
$$M = \frac{1}{2}\frac{\operatorname{tr}\left((U_{\lambda t} + [U_\lambda, V])\,[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}, \tag{32}$$
$$N = \frac{1}{2}\frac{\operatorname{tr}\left((V_{\lambda t} + [V_\lambda, V])\,[U_\lambda, V_\lambda]\right)}{\sqrt{\frac{1}{2}\operatorname{tr}([U_\lambda, V_\lambda]^2)}}. \tag{33}$$

So, we derive the equations for calculating the coefficients of 2FF and next step is to find it. For this purpose we find the commutators

$$[U_\lambda, U] = 2i\lambda^2\begin{pmatrix} 0 & -q_x \\ \bar{q}_x & 0 \end{pmatrix}, \tag{34}$$
$$[U_\lambda, V] = i\alpha\begin{pmatrix} 0 & -q_x \\ \bar{q}_x & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} -(|q|^2)_x & 0 \\ 0 & (|q|^2)_x \end{pmatrix} + 2\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{35}$$
$$[V_\lambda, V] = i(\alpha - \frac{1}{\lambda^2})\begin{pmatrix} 0 & -q_x \\ \bar{q}_x & 0 \end{pmatrix} + \frac{i}{\lambda}\begin{pmatrix} -(|q|^2)_x & 0 \\ 0 & (|q|^2)_x \end{pmatrix} + \beta\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{36}$$
$$[U_\lambda, V_\lambda] = \frac{i}{\lambda^3}\begin{pmatrix} 0 & -q_x \\ \bar{q}_x & 0 \end{pmatrix} - \frac{i}{2\lambda^2}\begin{pmatrix} -(|q|^2)_x & 0 \\ 0 & (|q|^2)_x \end{pmatrix} - \frac{2}{\lambda}\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{37}$$

where

$$\alpha = 2\lambda^2 + 2 - |q|^2 - \frac{1}{2\lambda^2},$$
$$\beta = 3 - \frac{1}{\lambda^2} + \frac{|q|^2}{2\lambda^2} - \frac{1}{4\lambda^4}.$$

Then, substituting the equations (34)-(37) into the equations (31)-(33) we can obtain

$$L = \frac{\frac{i}{2\lambda^2}(\bar{q}_x q_{xx} - q_x\bar{q}_{xx}) + 2|q_x|^2 - \bar{q}q_{xx} - q\bar{q}_{xx} + 2i\lambda^2(\bar{q}q_x - q\bar{q}_x)}{\sqrt{\frac{1}{\lambda^4}|q_x|^2 + \frac{2i}{\lambda^2}(\bar{q}q_x - q\bar{q}_x) - \frac{1}{4\lambda^2}((|q|^2)_x)^2 + 4|q|^2}}, \tag{38}$$

$$M = \frac{\frac{i}{2\lambda^2}(\bar{q}_x q_{xt} - q_x\bar{q}_{xt}) + \frac{\alpha}{\lambda^2}|q_x|^2 - ia(q\bar{q}_x - \bar{q}q_x) + \frac{1}{4\lambda^2}((|q|^2)_x)^2 - \bar{q}q_{xt} - q\bar{q}_{xt} - 4|q|^2}{\sqrt{\frac{1}{\lambda^4}|q_x|^2 + \frac{2i}{\lambda^2}(\bar{q}q_x - q\bar{q}_x) - \frac{1}{4\lambda^2}((|q|^2)_x)^2 + 4|q|^2}}, \tag{39}$$

$$N = \frac{M + \frac{1}{4\lambda^4}(q_t\bar{q}_x + \bar{q}_t q_x) - \frac{1}{\lambda^4}|q_x|^2 + \frac{i\beta}{2\lambda^2}(q\bar{q}_x - \bar{q}q_x) + \frac{1}{4\lambda^2}((|q|^2)_x)^2 + \frac{i}{2\lambda^2}(\bar{q}q_t - q\bar{q}_t) - 2(\beta - 2)|q|^2}{\sqrt{\frac{1}{\lambda^4}|q_x|^2 + \frac{2i}{\lambda^2}(\bar{q}q_x - q\bar{q}_x) - \frac{1}{4\lambda^2}((|q|^2)_x)^2 + 4|q|^2}}, \tag{40}$$

where $a = \alpha - 1/\lambda^2$.

Thus, we found a 2FF surface, which is defined by the equation (25), where the coefficients are equal to the expressions (38)-(40).

4. Surface area

If a surface in a Euclidean space is given by a parametrically smooth function $r(x, t)$, where the parameters x, t change in the D domain on the x, t plane, then the surface area S can be expressed by a double integral

$$S = \int\int_D |\mathbf{r}_x \wedge \mathbf{r}_t| dxdt, \tag{41}$$

where the module the vector product of the vectors $\mathbf{r}_x$ and $\mathbf{r}_t$ is equal to

$$|\mathbf{r}_x \wedge \mathbf{r}_t| = \frac{1}{2}\,\mathrm{tr}(r_{xx}).$$

It is known that r_{xx} is calculated by the formula (28), then the surface area S is calculated by the following formula:

$$S = \int\int_D \sqrt{\frac{1}{2}\,\mathrm{tr}\,(U_{\lambda x} + [U_\lambda, U])^2} dxdt, \tag{42}$$

then our required surface area S look as

$$S = \int\int_D \sqrt{|q_{xx}|^2 + 2i\lambda^2(\bar{q}_x q_{xx} - q_x\bar{q}_{xx}) + 4\lambda^4|q_x|^2} dxdt. \tag{43}$$

5. Gaussian surface curvature

The Gaussian curvature of a surface is the product of the principal curvatures of a regular surface at a given point and calculate by the ratio of the discriminants of the first and second fundamental forms [15]

$$K = \frac{LN - M^2}{EG - F^2}, \tag{44}$$

that is, by substituting equations (38)-(40) and (19)-(21) into the equation (44), we can obtain the Gaussian curvature.

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6. Conclusion

Thus, in this paper, for the Fokas-Lenells equation using the Sym-Tafel formula, 1FF and 2FF surfaces with corresponding coefficients are found. 2FF is an effective tool for studying the geometric properties of a regular surface. Through this form, can enter important geometric characteristics that measure the degree and type of surface deviation from the tangent plane. Using these forms, gave a formula for calculating the Gaussian curvature of a surface, which is convenient to use for the classification of points of a regular surface: the sign of which at a given point indicates the nature of the surface behavior at this point ($K > 0$ is elliptic, $K < 0$ is hyperbolic, $K = 0$ is parabolic). Also found the surface area.

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