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Exact Solution of the Nonlocal $\mathcal{PT}$-Symmetric (2 + 1)-Dimensional Hirota–Maxwell–Bloch System

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Abstract: This paper investigates the (2 + 1)-dimensional nonlocal Hirota–Maxwell–Bloch (NH-MB) system under various types of nonlocality. The mathematical consistency of possible nonlocal structures is analyzed, and three types that lead to a well-posed system are identified. The integrability of the system is established through its Lax pair representation, and a Darboux transformation is constructed. Exact soliton solutions are obtained for both the defocusing and focusing cases. The results obtained may find applications in nonlinear optics, quantum theory, and the theory of integrable systems.

Keywords: nonlocal system; $\mathcal{PT}$-symmetry; Hirota–Maxwell–Bloch equation; Darboux transformation

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1. Introduction

Nonlocal integrable equations represent a modern and actively developing field in the theory of nonlinear dynamical systems and mathematical physics. Such models generalize classical nonlinear equations by incorporating space–time nonlocality, which allows for the adequate description of a wide range of complex physical phenomena—from nonlinear waves and soliton interactions to the generation of rogue waves in media with $\mathcal{PT}$-symmetry [1,2].

A significant contribution to the development of the theory of nonlocal integrable equations was made by the works of Ablowitz and Musslimani, who first proposed the nonlocal nonlinear Schrödinger equation and laid the foundation for constructing a broad class of $\mathcal{PT}$-symmetric models [3,4]. These studies became a starting point for subsequent research aimed at developing multidimensional and multicomponent nonlocal generalizations, including versions of the derivative nonlinear Schrödinger and KdV equations [5–7]. Recent results cover a wide spectrum of nonlinear structures—from classical solitons and rogue waves to complex quasi-periodic and mixed waves [8–10].

Among multicomponent nonlocal models, the integrable Hirota–Maxwell–Bloch (HMB) system holds a special place, widely used to describe nonlocal effects in quantum optics, nonlinear spin chains, and laser physics [11–13]. Despite significant achievements in the study of (1 + 1)-dimensional versions of the HMB system, its (2 + 1)-dimensional nonlocal generalizations with space–time and $\mathcal{PT}$-symmetric reductions, as well as issues related to the consistency of reductions and the dynamics of nonlocal soliton solutions, remain insufficiently explored.

The relevance of this study is driven by the need for an in-depth analysis of (2 + 1)-dimensional nonlocal integrable systems and the investigation of how space–time nonlocality affects the parameters and dynamics of nonlinear waves and soliton structures in multidimensional physical models. This is of particular importance for problems in nonlinear optics and quantum information, where multidimensional nonlocal systems are used to model signal propagation in metamaterials, waveguide arrays, feedback media, and to describe pulse interactions in complex nonlinear environments.

Much attention in contemporary literature is devoted to the development of analytical methods for solving nonlocal systems. Among the most effective approaches are the inverse scattering transform, the Riemann–Hilbert method, the subsidiary equation method, and techniques based on hyperbolic functions, which are widely used to construct exact solutions of various nonlocal models [14–18].

Of particular interest in the context of this work is the Darboux transformation method, which enables the construction of both single- and multi-soliton solutions without resorting to numerical schemes while preserving full integrability of the system [19–21]. In this paper, the Darboux transformation method is applied to a (2 + 1)-dimensional nonlocal $\mathcal{PT}$-symmetric Hirota–Maxwell–Bloch system. This approach allows for the efficient construction of exact solutions and the analysis of parameters of nonlocal structures in both focusing and defocusing regimes. However, it is important to consider the method's limitations related to the choice of consistent reductions and the symmetry properties of the system.

The main contribution of this work is the complete classification of consistent (2 + 1)-dimensional $\mathcal{PT}$-symmetric reductions of the Hirota–Maxwell–Bloch system and the construction of new exact one-soliton solutions for each type of nonlocality.

This research complements and extends previous results obtained for the one-dimensional nonlocal HMB system [12,13], and is consistent with more general studies of nonlocal multidimensional Schrödinger-type systems [17,19].

Unlike the (1 + 1)-dimensional model, where temporal nonlocality is defined by the condition

$$r(x,t) = \sigma q^*(x,-t), \quad \sigma = \pm 1,$$

the multidimensional system allows for more complex spatial, temporal, and space–time reductions [13]. These types of $\mathcal{PT}$-symmetric constraints significantly influence the structure and dynamic behavior of the solutions.

The aim of this work is the analytical study of the (2 + 1)-dimensional Hirota–Maxwell–Bloch system using methods from the theory of integrable systems [11,12]. The system under consideration takes the following form

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} + i(wq)_x - vq - 2ip = 0, \tag{1}$$
$$ir_t - \epsilon_1 r_{xy} + i\epsilon_2 r_{xxy} + i(wr)_x + vr - 2i\kappa = 0, \tag{2}$$
$$v_x + 2\epsilon_1 (qr)_y - 2i\epsilon_2 (qr_{xy} - rq_{xy}) = 0, \tag{3}$$
$$w_x - 2\epsilon_2 (qr)_y = 0, \tag{4}$$
$$p_x - 2i\omega p - 2q\eta = 0, \tag{5}$$
$$\kappa_x + 2i\omega\kappa - 2r\eta = 0, \tag{6}$$
$$\eta_x + (rp + \kappa q) = 0, \tag{7}$$

where the functions $q(x,y,t)$, $r(x,y,t)$, $p(x,y,t)$, and $\kappa(x,y,t)$ are complex-valued, while $v(x,y,t)$, $w(x,y,t)$, and $\eta(x,y,t)$ are real-valued. Here, ϵ_1 and ϵ_2 are certain parameters, while δ and ω are real constants, with ω representing the frequency. The variable λ is the spectral parameter.

This work consists of six sections. Section 2 presents the classification of consistent $\mathcal{PT}$-symmetric reductions. Section 3 provides the Lax pair formulation and proves the integrability of the model. Section 4 is devoted to constructing the one-fold Darboux transformation. Section 5 presents one-soliton solutions and their parametric analysis. Section 6 provides an analysis of the obtained solutions. The Conclusion discusses the results and outlines possible directions for future research.

2. Classification and Analysis of Nonlocal Generalizations of the (2 + 1)-Dimensional Hirota–Maxwell–Bloch System

The (2 + 1)-dimensional Hirota–Maxwell–Bloch (HMB) system is integrable within the framework of the inverse scattering method [22]. In this work, we examine various types of nonlocal generalizations of this system and their compatibility with the dynamical properties of the original model.

Let us first consider one of the non-integrable cases.

Non-integrable case: reflection in x and y

We consider a $\mathcal{PT}$-symmetric nonlocality with reflection in both spatial variables x, y:

$$r(x,\, y,\, t) = \sigma q^*(-x,\, -y,\, t), \quad \kappa(x,\, y,\, t) = \delta p^*(-x,\, -y,\, t). \tag{8}$$

Substituting these conditions (8) into the original Equations (1) and (2), we obtain:

$$\begin{aligned}
&iq_t(x,y,t) + \epsilon_1 q_{xy}(x,y,t) + i\epsilon_2 q_{xxy}(x,y,t) + i[w(x,y,t)q(x,y,t)]_x - \\
&v(x,y,t)q(x,y,t) - 2ip(x,y,t) = 0, \\
&i\sigma q_t^*(-x,-y,t) - \epsilon_1 \sigma q_{xy}^*(-x,-y,t) - i\epsilon_2 \sigma q_{xxy}^*(-x,-y,t) + \\
&i\sigma[w_x(x,y,t)q^*(-x,-y,t) - w(x,y,t)q_x^*(-x,-y,t)] + \\
&\sigma v(x,y,t)q^*(-x,-y,t) - 2i\delta p^*(-x,-y,t) = 0.
\end{aligned}$$

Applying the change of variables $x \to -x$, $y \to -y$ and complex conjugation yields:

$$\begin{aligned}
&iq_t(x,y,t) + \epsilon_1^* q_{xy}(x,y,t) - i\epsilon_2^* q_{xxy}(x,y,t) + \\
&i[w_x(-x,-y,t)q(x,y,t) - w(-x,-y,t)q_x(x,y,t)] - \\
&v(-x,-y,t)q(x,y,t) - 2i\delta\sigma p(x,y,t) = 0.
\end{aligned}$$

For the system to be consistent, the following conditions must be met:

$$\sigma = -\delta, \quad \epsilon_1 = \epsilon_1^*, \quad \epsilon_2 = -\epsilon_2^*, \quad \omega = -\omega^*. \tag{9}$$

Since ω is a real parameter, the last condition is not satisfied, making the system inconsistent.

Similarly, it can be shown that the following types of nonlocality also lead to contradictions:

- Reflection in variables x and t:

$$r(x,y,t) = \sigma q^*(-x,y,-t), \quad \kappa(x,y,t) = \delta p^*(-x,y,-t).$$

- Reflection in all variables x, y, t:

$$r(x,y,t) = \sigma q^*(-x,-y,-t), \quad \kappa(x,y,t) = \delta p^*(-x,-y,-t).$$

- Reflection in the variable x:

$$r(x,y,t) = \sigma q^*(-x,y,t), \quad \kappa(x,y,t) = \delta p^*(-x,y,t).$$

Now, let us consider the consistent types of nonlocal reductions.
Case 1: reflection with respect to the variable y
We impose the nonlocal condition with reflection in y:

$$r(x,\, y,\, t) = \sigma q^*(x,\, -y,\, t), \quad \kappa(x,\, y,\, t) = \delta p^*(x,\, -y,\, t). \tag{10}$$

Substituting this into the NH-MB system results in a consistent system of equations:

$$\begin{aligned}
iq_t(x,\, y,\, t) \quad &+ \quad \epsilon_1 q_{xy}(x,\, y,\, t) + i\epsilon_2 q_{xxy}(x,\, y,\, t) + i[w(x,\, y,\, t)q(x,\, y,\, t)]_x - \\
&v \quad (x,\, y,\, t)q(x,\, y,\, t) - 2ip(x,\, y,\, t) = 0, &(11)\\
v_x(x,\, y,\, t) \quad &+ \quad 2\epsilon_1\sigma[q_y(x,\, y,\, t)q^*(x,\, -y,\, t) - q(x,\, y,\, t)q_y^*(x,\, -y,\, t)] + \\
&2 \quad i\epsilon_2\sigma[q(x,\, y,\, t)q_{xy}^*(x,\, -y,\, t) + q^*(x,\, -y,\, t)q_{xy}(x,\, y,\, t)] = 0, &(12)\\
w_x(x,\, y,\, t) \quad &- \quad 2\epsilon_2\sigma[q_y(x,\, y,\, t)q^*(x,\, -y,\, t) - q(x,\, y,\, t)q_y^*(x,\, -y,\, t)] = 0, &(13)\\
p_x(x,\, y,\, t) \quad &- \quad 2i\omega p(x,\, y,\, t) - 2q(x,\, y,\, t)\eta(x,\, y,\, t) = 0, &(14)\\
\eta_x(x,\, y,\, t) \quad &+ \quad \sigma p(x,\, y,\, t)q^*(x,\, -y,\, t) + \delta p^*(x,\, -y,\, t)q(x,\, y,\, t), &(15)
\end{aligned}$$

Case 2: reflection with respect to the temporal variable t
We impose the nonlocal condition with reflection in t:

$$r(x,y,t) = \sigma q^*(x,y,-t), \quad \kappa(x,y,t) = \delta p^*(x,y,-t). \tag{16}$$

The system becomes:

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} + i[wq]_x - vq - 2ip = 0, \tag{17}$$
$$v_x + 2\epsilon_1\sigma[qq^*]_y - 2i\epsilon_2\sigma(qq_{xy}^* - q^* q_{xy}) = 0, \tag{18}$$
$$w_x - 2\epsilon_2\sigma[qq^*]_y = 0, \tag{19}$$
$$p_x - 2i\omega p - 2q\eta = 0, \tag{20}$$
$$\eta_x + \sigma pq^* + \delta p^* q = 0. \tag{21}$$

Case 3: reflection with respect to y and t
We impose the nonlocal condition with reflection in both y and t:

$$r(x,y,t) = \sigma q^*(x,-y,-t), \quad \kappa(x,y,t) = \delta p^*(x,-y,-t). \tag{22}$$

The system now reads:

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} + i[wq]_x - vq - 2ip = 0, \tag{23}$$
$$v_x + 2\epsilon_1\sigma(q_y q^* - qq_y^*) + 2i\epsilon_2\sigma(qq_{xy}^* + q^* q_{xy}) = 0, \tag{24}$$
$$w_x - 2\epsilon_2\sigma(q_y q^* - qq_y^*) = 0, \tag{25}$$
$$p_x - 2i\omega p - 2q\eta = 0, \tag{26}$$
$$\eta_x + \sigma pq^* + \delta p^* q = 0. \tag{27}$$

Each equation in the considered systems plays a crucial role in maintaining their consistency and integrability. The wave function $q(x,y,t)$ governs the nonlinear dynamics and interacts with the potentials $v(x,y,t)$ and $p(x,y,t)$, shaping the overall structure of the model. The potential $v(x,y,t)$ defines the nonlocal links with $q(x,y,t)$, while the function $w(x,y,t)$ provides an additional coupling with the main field. The dynamics of $p(x,y,t)$ are determined by its interaction with $\eta(x,y,t)$, and the equation for $\eta(x,y,t)$ closes the system and ensures consistency under the given symmetry constraints.

The analysis shows that only three types of nonlocality preserve the consistency of the Hirota–Maxwell–Bloch system:

$$r(x,y,t) = \sigma q^*(x,-y,t), \quad r(x,y,t) = \sigma q^*(x,-y,-t), \quad r(x,y,t) = \sigma q^*(x,y,-t), \tag{28}$$

which allows for the construction of self-consistent solutions.

Further attention is focused on the case with reflection in the variable y:

$$r(x,y,t) = \sigma q^*(x,-y,t), \tag{29}$$

for which the system of equations is constructed and analytical methods, including the study of soliton solutions, are proposed.

The different types of nonlocal reductions and their consistency conditions are summarized in Table 1. In the next section, we proceed to construct the Lax pair for this type of nonlocality.

Table 1. Classification of nonlocality types and system consistency.

Nonlocality Type	Reflection	Consistency	Conditions
$\sigma q^*(-x,-y,t)$	$x \to -x, y \to -y$	No	$\omega \neq -\omega$
$\sigma q^*(-x,y,-t)$	$x \to -x, t \to -t$	No	Conflict in v, w
$\sigma q^*(-x,-y,-t)$	$x,y,t \to -x,-y,-t$	No	Conflict in v, w
$\sigma q^*(-x,y,t)$	$x \to -x$	No	Conflict in v, w
$\sigma q^*(x,-y,t)$	$y \to -y$	Yes	$\sigma = \delta, \omega \in \mathbb{R}$
$\sigma q^*(x,y,-t)$	$t \to -t$	Yes	$\sigma = -\delta, \omega \in \mathbb{R}$
$\sigma q^*(x,-y,-t)$	$y \to -y, t \to -t$	Yes	$\sigma = -\delta, \omega \in \mathbb{R}$

3. Lax Representation of the (2 + 1)-Dimensional Nonlocal Hirota–Maxwell–Bloch System

One of the key methods for studying integrable nonlinear equations is the Lax representation, which plays a fundamental role in the theory of integrable systems. Its existence not only confirms the integrability of the given system but also enables the application of powerful analytical methods, such as the inverse scattering transform and Darboux transformations, to obtain exact solutions.

In this section, we consider the Lax representation of the (2 + 1)-dimensional nonlocal Hirota–Maxwell–Bloch system, taking into account the nonlocal condition (10). Constructing the Lax pair is a crucial step in investigating this system, as it allows the nonlinear system (11)–(15) to be associated with a linear problem, thereby facilitating the constructive search for solutions.

The spectral Lax equations for the considered nonlocal system take the following form:

$$\varphi_x(x,\, y,\, t;\, \lambda) = A(x,\, y,\, t;\, \lambda)\varphi(x,\, y,\, t;\, \lambda), \tag{30}$$

$$\varphi_t(x,\, y,\, t;\, \lambda) = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\varphi_y(x,\, y,\, t;\, \lambda) + B(x,\, y,\, t;\, \lambda)\varphi(x,\, y,\, t;\, \lambda). \tag{31}$$

Here, φ is a vector function, and $A(x,\, y,\, t;\, \lambda)$ and $B(x,\, y,\, t;\, \lambda)$ are matrices depending on the spectral parameter λ. For the system to be integrable, the compatibility condition of these equations must hold, leading to the zero-curvature equation:

$$A_t - B_x - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)A_y + [A, B] = 0. \tag{32}$$

Here, $[A, B]$ denotes the commutator of the two matrices A and B. This zero-curvature equation ensures the existence of a consistent solution to Equations (30) and (31), thereby confirming the integrability of the nonlocal Hirota–Maxwell–Bloch system.

Lemma 1. *The Lax representation of the (2 + 1)-dimensional nonlocal Hirota–Maxwell–Bloch system* (30) *and* (31) *is consistent if and only if the zero-curvature Equation* (32) *holds.*

Proof. The necessary and sufficient condition for the consistency of the linear system (30) and (31) is that their mixed derivatives with respect to x and t coincide:

$$\varphi_{xt} = \varphi_{tx}. \tag{33}$$

Substituting expressions (30) and (31), we obtain

$$A_t\varphi + A\varphi_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\varphi_{yx} + B_x\varphi + B\varphi_x. \tag{34}$$

Using (30) and (31) again, we arrive at the equation

$$(A_t - B_x - (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)A_y + [A, B])\varphi = 0. \tag{35}$$

Since φ is an arbitrary function, Equation (32) must hold. This proves the lemma.

The matrix $A(x, y, t; \lambda)$ has the form:

$$A = -i\lambda\sigma_3 + A_0, \tag{36}$$

where λ is the spectral parameter, σ_3 is one of the Pauli matrices, and A_0 is a matrix independent of λ:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & q(x, y, t) \\ -\sigma q^*(x, -y, t) & 0 \end{pmatrix}. \tag{37}$$

The matrix $B(x, y, t; \lambda)$ is represented as follows:

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}, \tag{38}$$

where the elements of the matrix $B(x, y, t; \lambda)$ are given by:

$$B_1 = iw\sigma_3 + 2i\epsilon_2\sigma_3 A_{0y} = \begin{pmatrix} iw(x, y, t) & 2i\epsilon_2 q_y(x, y, t) \\ -2i\epsilon_2\sigma q_y^*(x, -y, t) & -iw(x, y, t) \end{pmatrix}, \tag{39}$$

$$B_0 = \begin{pmatrix} -\frac{i}{2}v & i\epsilon_1 q_y - \epsilon_2 q_{xy} - wq \\ -i\epsilon_1\sigma q_y^* - \epsilon_2\sigma q_{xy}^* + \sigma w q^* & \frac{i}{2}v \end{pmatrix}, \tag{40}$$

$$B_{-1} = \begin{pmatrix} \eta(x, y, t) & -p(x, y, t) \\ -2\delta p^*(x, -y, t) & -\eta(x, y, t) \end{pmatrix}. \tag{41}$$

Thus, the existence of the Lax representation guarantees the integrability of the nonlocal Hirota–Maxwell–Bloch system (11)–(15). This confirms the possibility of solving it using inverse scattering methods. This system exhibits all the characteristic properties of integrable equations, such as an infinite number of conservation laws and the presence of soliton solutions. □

The studied system describes the propagation of optical pulses in nonlinear media, such as erbium-doped optical fibers. It accounts for nonlinear effects, higher-order dispersion, and the interaction of the optical field with the atomic medium. The application of nonlocal conditions in such systems allows for the inclusion of feedback effects, which can be useful for modeling new types of optical waves.

In the next section, we will consider the application of the Darboux transformation method to obtain exact solutions for the nonlocal Hirota–Maxwell–Bloch system using the Lax pair structure defined in this section.

4. Darboux Transformation for the Nonlocal Hirota–Maxwell–Bloch System

The Darboux transformation is a powerful tool for obtaining exact solutions of integrable nonlinear systems, including nonlocal equations. It not only generates new solutions from known ones but also preserves the integrable structure of the system. In this section, we present the first-order Darboux transformation for the $\mathcal{PT}$-symmetric nonlocal NH-MB system.

Consider the following setup. Let φ be a fundamental solution of the Lax pair given by Equations (30) and (31).

The Darboux transformation is constructed in the form of a linear transformation:

$$\varphi' = T\varphi, \quad T = \lambda I - S, \tag{42}$$

where T is the Darboux transformation matrix, depending on the spectral parameter λ, I is the 2×2 identity matrix, and S is a matrix containing unknown elements s_{kj}, determined by the compatibility conditions:

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}. \tag{43}$$

The elements of matrix S are functions of the variables x, y, t and must be determined such that the new Lax pair matrices preserve the compatibility conditions, ensuring the integrability of the system.

Applying the Darboux transformation to the system (30) and (31) leads to a new Lax pair:

$$\varphi'_x = A'\varphi', \tag{44}$$

$$\varphi'_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\varphi'_y + B'\varphi', \tag{45}$$

where A' and B' are the new matrices corresponding to the transformed system.

Theorem 1. *Let $\varphi' = T\varphi$ be the new solution of the Lax pair, and let the Darboux transformation matrix be given by $T = \lambda I - S$. Then, the new potentials q', v', w', p', η' of the nonlocal NH-MB system are expressed in terms of the original potentials q, v, w, p, η as follows:*

$$q' = q - 2is_{12}, \tag{46}$$

$$w' = w - 4i\epsilon_2 s_{11y}, \tag{47}$$

$$v' = v + 4i\epsilon_1 s_{11y} - 4\epsilon_2(\sigma s_{12}q_y^* + s_{21}q_y - 2is_{11}s_{11y} - 2is_{21}s_{12y}), \tag{48}$$

$$\eta' = \frac{[(s_{11}+\omega)(s_{22}+\omega) + s_{12}s_{21}]\eta - \delta s_{12}(s_{22}+\omega)p^* + \delta(s_{11}+\omega)s_{21}p}{(s_{11}+\omega)(s_{22}+\omega) - s_{12}s_{21}}, \tag{49}$$

$$p' = \frac{2s_{12}(s_{11}+\omega)\eta - \delta s_{12}^2 p^* + (s_{11}+\omega)^2 p}{(s_{11}+\omega)(s_{22}+\omega) - s_{12}s_{21}}. \tag{50}$$

Proof. Substituting the Darboux transformation (44) and (45) into the system equations, we obtain the compatibility conditions:

$$T_x + TA = A'T, \tag{51}$$
$$T_t + TB = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)T_y + B'T. \tag{52}$$

Considering different powers of λ, we derive equations for the components of S, which are then used to express the new potentials in terms of the original ones. Detailed calculations lead to formulas (46)–(50).

Furthermore, we express S through eigenvectors:

$$S = H\Lambda H^{-1}, \tag{53}$$

where

$$H = \begin{pmatrix} \varphi_{1,1}(x,y,t) & \sigma\varphi_{2,1}^*(x,y,t) \\ \varphi_{2,1}(x,y,t) & \varphi_{1,1}^*(x,y,t) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{54}$$

where λ_1 is a complex number, and $\lambda_2 = \lambda_1^*$. $\mathcal{PT}$-symmetry for the eigenvectors $\varphi_{k,j}$ $(k, j = 1, 2)$ imposes the following conditions:

$$\varphi_{1,2}(x,y,t) = \sigma\varphi_{2,1}^*(x,-y,t), \quad \varphi_{2,2}(x,y,t) = \varphi_{1,1}^*(x,-y,t). \tag{55}$$

Using this representation, we express the new solutions in terms of the initial functions:

$$q' = q + \frac{2i\sigma(\lambda-\lambda^*)\varphi_{1,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}, \tag{56}$$

$$w' = w - 4i\epsilon_2\left[\frac{\lambda_1\varphi_{1,1}\varphi_{1,1}^* - \sigma\lambda_1^*\varphi_{2,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\right]_y, \tag{57}$$

$$\begin{aligned} v' = v &+ 4i\epsilon_1\left[\frac{\lambda_1\varphi_{1,1}\varphi_{1,1}^* - \sigma\lambda_1^*\varphi_{2,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\right]_y + \frac{4\epsilon_2(\lambda_1-\lambda_1^*)\varphi_{1,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\cdot q_y^* - \\ &\frac{4\epsilon_2(\lambda_1-\lambda_1^*)\varphi_{1,1}^*\varphi_{2,1}}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\cdot q_y + 8i\epsilon_2\frac{\lambda_1\varphi_{1,1}\varphi_{1,1}^* - \sigma\lambda_1^*\varphi_{2,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\cdot\left[\frac{\lambda_1\varphi_{1,1}\varphi_{1,1}^* - \sigma\lambda_1^*\varphi_{2,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\right]_y - \\ &8i\sigma\epsilon_2\frac{(\lambda_1-\lambda_1^*)\varphi_{1,1}^*\varphi_{2,1}}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\cdot\left[\frac{(\lambda_1-\lambda_1^*)\varphi_{1,1}\varphi_{2,1}^*}{\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*}\right]_y, \end{aligned} \tag{58}$$

$$\begin{aligned} \eta' = &\frac{\sigma}{\Delta_1}\{p^*\cdot(\lambda_1-\lambda_1^*)\cdot\varphi_{1,1}\varphi_{2,1}^*\cdot[(\omega+\lambda_1^*)\varphi_{1,1}\varphi_{1,1}^* - \sigma(\omega+\lambda_1)\varphi_{2,1}\varphi_{2,1}^*] + \\ &p\cdot(\lambda_1-\lambda_1^*)\varphi_{1,1}^*\varphi_{2,1}\cdot[(\omega+\lambda_1)\varphi_{1,1}\varphi_{1,1}^* - \sigma(\omega+\lambda_1^*)\varphi_{2,1}\varphi_{2,1}^*]\} + \\ &\eta\cdot\left[1 - \frac{2}{\Delta_1}(\lambda_1-\lambda_1^*)^2\varphi_{1,1}\varphi_{1,1}^*\varphi_{2,1}\varphi_{2,1}^*\right], \end{aligned} \tag{59}$$

$$\begin{aligned} p' = &\frac{1}{\Delta_1}\{-\sigma p^*\cdot[(\lambda_1-\lambda_1^*)\cdot\varphi_{1,1}\varphi_{2,1}^*]^2 + p\cdot[(\omega+\lambda_1)\varphi_{1,1}\varphi_{1,1}^* - \sigma(\omega+\lambda_1^*)\varphi_{2,1}\varphi_{2,1}^*]^2 - \\ &2\eta\cdot(\lambda_1-\lambda_1^*)\varphi_{1,1}\varphi_{2,1}^*\cdot[(\omega+\lambda_1)\varphi_{1,1}\varphi_{1,1}^* - \sigma(\omega+\lambda_1^*)\varphi_{2,1}\varphi_{2,1}^*]\}, \end{aligned} \tag{60}$$

where

$$\Delta_1 = (\omega+\lambda_1)(\omega+\lambda_1^*)(\varphi_{1,1}\varphi_{1,1}^* - \sigma\varphi_{2,1}\varphi_{2,1}^*)^2 \neq 0. \tag{61}$$

The Darboux transformation preserves the integrable structure of the system and is consistent with $\mathcal{PT}$-symmetry. It enables the derivation of exact analytical solutions, including soliton solutions, making it a powerful tool for analyzing nonlocal nonlinear systems. In the next section, we will explore the application of these solutions to the study of nonlocal solitons. □

5. One-Soliton Solutions

Applying the Darboux transformation (56)–(60), we obtain one-soliton solutions. Let us introduce the notation:

$$\lambda_1 = \alpha_1 + i\beta_1, \quad \mu_1 = \gamma_1 + i\tau_1, \quad \alpha_1, \beta_1, \gamma_1, \tau_1 \in \mathbb{R}. \tag{62}$$

For the defocusing $\mathcal{PT}$-symmetric nonlocal HMB system ($\sigma = 1$), the one-soliton solutions take the form:

$$q' = -2\beta_1 e^{f_1} \operatorname{csch}(f_2), \tag{63}$$

$$w' = -4\beta_1 \epsilon_2 f_{2y} \operatorname{csch}^2(f_2), \tag{64}$$

$$v' = 4\beta_1 \operatorname{csch}^2(f_2)\{\epsilon_1 f_{2y} + 2\epsilon_2 f_{2y}[\alpha_1 + i\beta_1 \coth(f_2)] + 2i\epsilon_2\beta_1[f_{1y} - f_{2y}\coth(f_2)]\}, \tag{65}$$

$$\eta' = 1 + \frac{2\beta_1^2}{(\alpha_1+\omega)^2 + \beta_1^2} \operatorname{csch}^2(f_2), \tag{66}$$

$$p' = -\frac{2i\beta_1 e^{f_1}}{(\alpha_1+\omega)^2+\beta_1^2}\left[(\alpha_1+\omega)\operatorname{csch}(f_2) + i\beta_1 \frac{\operatorname{csch}^2(f_2)}{\operatorname{sech}(f_2)}\right], \tag{67}$$

where

$$f_1 = -2i\alpha_1 x - 2\tau_1 y + 4i\epsilon_2(\alpha_1\gamma_1 - \beta_1\tau_1)t + 8i\epsilon_2(\alpha_1^2\gamma_1 - \beta_1^2\gamma_1 - 2\alpha_1\beta_1\tau_1)t + \frac{2i(\alpha_1+\omega)t}{(\alpha_1+\omega)^2+\beta_1^2} + 2i\delta_2 - i\delta_0, \tag{68}$$

$$f_2 = 2\beta_1 x + 2i\gamma_1 y + 4\epsilon_1(\beta_1\gamma_1 + \tau_1\alpha_1)t - 8\epsilon_2(2\beta_1\gamma_1\alpha_1 + \alpha_1^2\tau_1 - \beta_1^2\tau_1)t + \frac{2\beta_1 t}{(\alpha_1+\omega)^2+\beta_1^2} + 2\delta_1. \tag{69}$$

For the focusing $\mathcal{PT}$-symmetric nonlocal HMB system ($\sigma = -1$), the solutions take the form:

$$q' = 2\beta_1 e^{f_1} \operatorname{sech}(f_2), \tag{70}$$

$$w' = 4\beta_1 \epsilon_2 f_{2y} \operatorname{sech}^2(f_2), \tag{71}$$

$$v' = -4\beta_1 \operatorname{sech}^2(f_2)\{\epsilon_1 f_{2y} + 2\epsilon_2 f_{2y}[\alpha_1 + i\beta_1 \tanh(f_2)] + 2i\epsilon_2\beta_1[f_{1y} - f_{2y}\tanh(f_2)]\}, \tag{72}$$

$$\eta' = 1 - \frac{2\beta_1^2}{(\alpha_1+\omega)^2+\beta_1^2} \operatorname{sech}^2(f_2), \tag{73}$$

$$p' = -\frac{2i\beta_1 e^{f_1}}{(\alpha_1+\omega)^2+\beta_1^2}\left[(\alpha_1+\omega)\operatorname{sech}(f_2) + i\beta_1 \frac{\operatorname{sech}^2(f_2)}{\operatorname{csch}(f_2)}\right], \tag{74}$$

Thus, we have derived exact one-soliton solutions for the $\mathcal{PT}$-symmetric nonlocal Hirota–Maxwell–Bloch (HMB) system in both defocusing ($\sigma = 1$) and focusing ($\sigma = -1$)

regimes. The solutions are expressed via hyperbolic cosecant and hyperbolic secant functions and exhibit a rich structure depending on the model parameters.

These formulas allow us to study key characteristics of nonlocal solitons, such as their amplitude-phase modulation, width, and dynamics in space and time. The results provide a foundation for further qualitative analysis and interpretation of nonlocality effects in integrable systems.

In the next section, we will present a detailed physical interpretation of the obtained solutions, analyze the influence of parameters on their structure and dynamics, and compare the results with previously published studies on nonlocal integrable systems.

6. Analysis and Discussion of the Obtained Solutions

This section presents a detailed analysis of the one-soliton solutions obtained for the $\mathcal{PT}$-symmetric nonlocal Hirota–Maxwell–Bloch (NH-MB) system with time reflection. We will discuss the choice of the solution type, the influence of parameters, the physical interpretation, and also compare our results with previously published works on nonlocal integrable systems.

6.1. Choice of One-Soliton Solution and Its Features

One-soliton solutions were selected as fundamental structures that play a key role in the theory of integrable systems. They clearly demonstrate essential nonlocal effects, such as interactions with the nonlocal background and the influence of $\mathcal{PT}$-symmetry on the dynamics of solutions.

Two types of one-soliton solutions are considered in this study:

- For the defocusing regime ($\sigma = 1$)—bright solitons with a $\operatorname{csch}(f_2)$ profile, characterized by localization and exponential decay at infinity;
- For the focusing regime ($\sigma = -1$)—dark solitons with a $\operatorname{sech}(f_2)$ profile, featuring a non-zero asymptotic background and a localized amplitude dip.

One-soliton solutions also serve as a crucial step for constructing multi-soliton solutions through iterative application of the Darboux transformation, making them a justified choice for the primary analysis of the nonlocal NH-MB system.

6.2. Case Study: Soliton Structure for Specific Parameters

For the defocusing case ($\sigma = 1$) with parameters:

$$\alpha_1 = 1, \quad \beta_1 = 0.5, \quad \gamma_1 = 0, \quad \tau_1 = 0, \quad \omega = 0.5,$$

the profile $q'(x, y, t)$ corresponds to a bright soliton localized along the x-axis with a hyperbolic cosecant shape and weak modulation along y. This solution describes a single wave with a distinct peak and exponential decay.

In the focusing regime ($\sigma = -1$) with the same parameters, the soliton takes the form of a hyperbolic secant and corresponds to a dark soliton structure, which is typical for models with focusing interaction and $\mathcal{PT}$-symmetry.

6.3. Influence of Model Parameters

The dynamics and shape of the solutions are strongly affected by the choice of parameters:

- Spectral parameters α_1, β_1:
  - β_1 controls the width and localization of the soliton;
  - α_1 determines the phase velocity along the x-axis.
- Parameters γ_1 and τ_1 influence spatial modulation along the y-axis.

- Dispersion parameters ϵ_1 and ϵ_2 regulate the contribution of higher-order nonlinearity and nonlocal interactions.
- Parameter ω is related to the resonance properties of the medium and affects the amplitude of the solutions.
- Phase shifts δ_0, δ_1, δ_2 allow modeling of initial conditions and influence the spatial positioning of solitons.

6.4. Physical Interpretation and the Role of Nonlocality

The obtained solutions demonstrate the influence of nonlocal $\mathcal{PT}$-symmetry conditions on wave dynamics:

- Phase shifts arise due to nonlocal field reflections.
- Amplitude modulations and spatial asymmetries appear in the soliton profile, which are absent in classical local systems.
- The solutions show sensitivity to the parameter ϵ_2, responsible for higher-order nonlocality.

These effects are important for applications in optics (e.g., $\mathcal{PT}$-symmetric waveguides) and nonlinear condensed matter physics.

6.5. Comparative Analysis with Known Results

Thus, in contrast to previous studies, this work presents, for the first time, the analysis of a $(2+1)$-dimensional nonlocal HMB system with space–time $\mathcal{PT}$-symmetry. A comparison with earlier studies on related local and nonlocal integrable systems is presented in Table 2.

Table 2. Comparison with other nonlocal integrable systems.

Authors/Model	Features	Methods and Solutions
M. J. Ablowitz, Z. H. Musslimani (2013) [3]	Nonlocal nonlinear Schrödinger equation (1 + 1) with time reflection symmetry	Inverse scattering transform, bright and dark soliton solutions
R. Yesmakhanova et al. (2017) [11]	Local (2 + 1)-dimensional Hirota–Maxwell–Bloch system	Darboux transformation, bright and dark solitons
X. F. Li, J. S. He, K. Porsezian (2013) [12]	Local (1 + 1)-dimensional Hirota–Maxwell–Bloch system	Hirota bilinear method, rogue waves and soliton solutions
N. Myrzakulova et al. (2024) [13]	Nonlocal (1 + 1)-dimensional Hirota–Maxwell–Bloch system with time-reflection symmetry	Darboux transformation, bright solitons under nonlocal constraints
This work	Nonlocal (2 + 1)-dimensional $\mathcal{PT}$-symmetric Hirota–Maxwell–Bloch system	Lax pair, classification of reductions, bright and dark one-soliton solutions

7. Conclusions

In this work, we analyzed various types of nonlocal generalizations of the (2 + 1)-dimensional integrable Hirota–Maxwell–Bloch (HMB) system. It was established that, among the many possible reflection transformations, the system retains its consistency only for three types of nonlocality: reflection in the y-variable, reflection in t, and simultaneous reflection in both y and t. The study focused in detail on the case of reflection with respect to y, for which a Lax pair and a one-soliton solution were constructed. The obtained results confirm the integrability of the chosen nonlocal system and demonstrate the possibility of the existence of solitonic structures.

The conducted analysis made it possible to identify symmetry conditions for the potential functions of the system and to establish important restrictions on the model parameters required for its mathematical consistency.

Future research directions

The results presented in this paper open several avenues for further development:

- Extension to other integrable systems: Investigation of nonlocal generalizations for (2 + 1)-dimensional systems such as nonlocal Davey–Stewartson-type equations, nonlocal sine-Gordon models, and models with double nonlocality.
- Analysis of multi-soliton solutions: Based on the constructed Lax pairs, it is possible to further obtain and classify multi-soliton and interacting solutions, including the use of bilinear forms or the inverse scattering transform method.
- Connection with physical applications: Exploration of applications of nonlocal systems to modeling processes in nonlinear optics, spin chains with time-reversal symmetry, and wave interactions in plasmas and metamaterials.
- Numerical simulations: Development of numerical schemes to study the stability and dynamics of solutions under various nonlocality regimes.

Further development of these directions will deepen the understanding of nonlocal integrable systems and their applications in modern physical models.

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