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Existence and Uniqueness of the Viscous Burgers' Equation with the p-Laplace Operator

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Abstract: In this paper, we investigate the existence and uniqueness of solutions for the viscous Burgers' equation for the isothermal flow of power-law non-Newtonian fluids $\rho(\partial_t u + u\partial_x u) = \mu\partial_x\left(|\partial_x u|^{p-2}\partial_x u\right)$, augmented with the initial condition $u(0,x) = u_0$, $0 < x < L$, and the boundary condition $u(t,0) = u(t,L) = 0$, where ρ is the density, μ the viscosity, u the velocity of the fluid, $1 < p < 2$, $L > 0$, and $T > 0$. We show that this initial boundary problem has an unique solution in the Buchner space $L^2\left(0,T;{W_0}^{1,p}(0,1)\right)$ for the given set of conditions. Moreover, numerical solutions to the problem are constructed by applying the modeling and simulation package COMSOL Multiphysics 6.0 at small and large Reynolds numbers to show the images of the solutions.

Keywords: p-Laplacian; power-law non-Newtonian fluid model; existence and uniqueness; Burgers' equation; Bochner space; Sobolev space; COMSOL Multiphysics

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1. Introduction

Burgers' equation, first introduced by J.M. Burgers, is a fundamental model for various physical processes, including shock wave propagation and turbulence [1]. Originally formulated for Newtonian fluids, Burgers' equation has been extensively analyzed. However, its generalization to non-Newtonian fluids, specifically those described by the power-law model, has received comparatively less attention for the case when $1 < p < 2$ [2].

The study of non-Newtonian fluids remains a crucial research area due to their complex rheology and broad applications in engineering, biology, and industry. Semi-analytical methods have been beneficial in analyzing their flow behavior under various conditions. For example, Hayat et al. [3] examined the three-dimensional flow of an Eyring–Powell nanofluid over an exponentially stretching sheet, while Ershkov et al. [4] derived exact solutions for non-Newtonian rivulet flows on a heated surface, providing insights into fluid dynamics and heat transfer. Pukhnachev [5] contributed a mathematical model for viscoelastic Maxwell fluids, offering a theoretical framework for studying elasticity and viscosity effects. These studies highlight the value of combining theoretical and semi-analytical methods to capture the complexity of non-Newtonian fluids. Among the most interesting applications of Burgers' equation in engineering or mechanics, we mention non-Newtonian fluids in an expanding channel [6].

We employ a particular case of the generalized Burgers' equation, formulated by Wei and Jordan [7], which was used to analyze acoustic propagation in power-law fluids. Traveling wave solutions of this equation were derived by Wei and Borden [8].

Building on this foundation, our work focuses on the existence and uniqueness of weak solutions to the viscous Burgers' equation. This study is framed within the context of Sobolev and Bochner spaces, which provide a robust mathematical environment for addressing weak solutions and ensuring appropriate regularity conditions [9].

The existence of weak solutions for nonlinear PDEs such as Burgers' equation has been thoroughly studied for Newtonian fluids [10–13]. Bendaas [12] studied inviscid and viscid Burgers' equations for $p = 2$ with initial conditions in the half plane. We note that in [13], the result concerning the existence of solutions to Burgers' equations with $p = 2$ was established in the space $L^2(0, T; H_0^1(0, 1))$. However, extending these results to power-law non-Newtonian fluids presents additional challenges due to viscosity-related nonlinearities [14,15]. The unique solutions for the generalized Burgers' equation based on power-law fluids remain largely unexplored for the case when $p \neq 2$, creating a gap in the current literature that this paper aims to address. We determine the existence and uniqueness of solutions to Burgers equations with a power-law index $1 < p < 2$ in $L^2\left(0, T; {W_0}^{1,p}(0, 1)\right)$. We omit the case when $2 < p < \infty$, since the proofs are similar, and easier.

Wei and Jordan [7] established the following generalized Burgers' equation:

$$\rho(\partial_t u + u\partial_x u) = -P_x + \partial_x \left[\left(\mu_B + \frac{4}{3}\mu k |\partial_x u|^{m-1} \right) \partial_x u \right]$$

This was established as a model to study acoustic traveling waves in non-Newtonian fluids with the power-law index $m = p - 1, 0 < m < 1$.

In this paper, we study the well-posedness and numerical solutions of a special case of the above equation with the initial and boundary conditions:

$$\begin{cases} \rho(\partial_t u + u\partial_x u) = \mu\partial_x \left(|\partial_x u|^{p-2} \partial_x u \right), \\ u(0, x) = u_0(x), & x \in [0, L], \\ u(t, 0) = u(t, L) = 0, & t \in [0, T]. \end{cases}$$

By using the dimensionless variables $x^* = \frac{1}{L}x$, $u^* = \frac{1}{u_0}u$, and $t^* = \frac{u_0}{L}t$ in the PDE, we obtain the following problem, in which x^*, u^*, and t^* are still denoted by x, u, and t:

$$\begin{cases} \partial_t u + u\partial_x u = \frac{1}{Re}\partial_x \left(|\partial_x u|^{p-2} \partial_x u \right), \\ u(0, x) = u_0(x), & x \in [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T] \end{cases} \tag{1}$$

with $1 < p < 2$. In problem (1), the differential operator on the right-hand side of the PDE is called the p-Laplacian, Re stands for the Reynolds number, which is a dimensionless quantity determining the flow behavior based on the balance of inertial and viscous forces. The Reynolds number for this case is given by

$$Re = \frac{\rho L^{p-1}}{\mu u_0^{p-3}},$$

where L and u_0 represent the characteristic length and velocity, respectively (see, for example, ref. [15]). The Reynolds number significantly influences the stability of the fluid velocity u in (1). Our numerical simulations demonstrate this in Section 4. When $p = 2$, Equation (1) becomes a well-known Burgers' equation for a Newtonian fluid:

$$\begin{cases} \partial_t u + u\partial_x u = \frac{1}{Re}\partial_x^2 u, \\ u(0, x) = u_0(x), & x \in [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T]. \end{cases}$$

The well-posedness of the viscous Burgers' equation for Newtonian fluid has been studied intensively in the literature (see, for example, refs. [10,16,17]). To our knowledge, the well-posedness of Problem (1) has not been studied before. In our work, we prove the result concerning the existence, uniqueness, and regularity of a solution to Burgers' equation in (1).

Multiplying the equation of (1) by a test function $w \in W_0^{1,p}(0,1)$ and integrating formally using parts from 0 to 1, we convert the initial boundary value problem (1) to the following integral identity:

$$\int_0^1 \partial_t u w dx + \int_0^1 u \partial_x u w dx + \frac{1}{Re} \int_0^1 \left(|\partial_x u|^{p-2} \partial_x u \right) \partial_x w dx = 0 \tag{2}$$

where in all cases, $w \in W_0^{1,p}(0,1)$, $t \in (0,T)$, and $1 < p < 2$.

To present our main results, we use the Bochner space $L^2(0,T;W_0^{1,p}(0,1))$ [18], and state the definition of the solution as the following .

Definition 1. *For $1 < p < 2$, a function $u \in L^2\left(0,T;W_0{}^{1,p}(0,1)\right)$, for which $\partial_t u \in L^2(0,T;L^2(0,1))$ is said to be a weak solution of Problem (1) if it satisfies (2) for each $w \in W_0{}^{1,p}(0,1)$ and a.e. $t \in (0,T)$ and $u(0,x) = u_0$ for all $x \in (0,1)$.*

The main result of this work is gathered in the following theorem on the existence and uniqueness of the solution for initial-boundary value (1).

Theorem 1. *Let $u_0 \in W_0^{1,p}(0,1)$. Then, there exists a unique solution $u \in L^2\left(0,T;W_0{}^{1,p}(0,1)\right)$ of (1) so that $\partial_t u \in L^2(0,T;L^2(0,1))$ in the sense of Definition 1.*

In Section 2, we state some preliminary information that is needed to study the well-posedness of Problem (1), formulated in the weak form (2). Section 3 consists of two sections, Sections 3.1 and 3.2. We obtain the existence result in Section 3.1 and establish the uniqueness of the solution in Section 3.2. Moreover, in Section 4, we construct the numerical solutions for the various values of the Reynolds number Re by using the COMSOL PDE solver with some numerical manipulations to avoid singularity in $\partial_x\left(|\partial_x u|^{p-2}\partial_x u\right)$ at points where $\partial_x u = 0$. The purpose of the COMSOL construction is to illustrate the graphic image of the solution.

2. Preliminary Mathematical Considerations

In this section, we state some well-known facts that we will exploit to prove the main Theorem 1 of this work. Gronwall's inequality, Jensen's inequality, Sobolev embedding, and the Bounded Convergence Theorems are required to attain our goal.

Recall that $L^p(0,1)$ and $W^{1,p}(0,1)$ are the standard Lebesgue and Sobolev spaces, respectively, for $1 < p < \infty$. For any Banach space X, we define $L^p(0,T;X)$ to be the space of measurable functions $u : (0,T) \to X$ so that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p \, \mathrm{d}t \right)^{1/p} < \infty$$

for $1 < p < \infty$ and $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0<t<T} \|u\|_X < \infty$ if $p = \infty$. $L^p(0,T;X)$ is a Banach space [13]. The Sobolev space $W^{1,p}(0,1)$ consists of functions whose weak derivatives are integrable in $L^p(0,1)$. $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in $L^2(0,1)$; that is,

$$\langle f, g \rangle = \int_0^1 f \bar{g} dx.$$

In the following, we present several propositions that are well-established theorems in the literature, which will be used in Section 3.

Proposition 1 (The differential form of Gronwall's inequality, (p. 708, [19])).

(i) *Let $\eta(\cdot)$ be a non-negative, absolutely continuous function on $[0,T]$ which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are non-negative, summable functions on $[0,T]$. Then,

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $0 \leq t \leq T$.

(ii) *In particular, if*

$$\eta' \leq \phi\,\eta \text{ on } [0,T] \text{ and } \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0,T].$$

Proposition 2 (Poincaré's inequality, Theorem 3, (p. 279, [19])). *Assume that U is a bounded, open subset of $\mathbb{R}^n$. Suppose that $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then, we have the following estimate:*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

where for each $q \in [1, \frac{np}{n-p}]$, the constant C only depends on p, q, n, and U.

In particular, for all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}. \tag{3}$$

Next is the Bounded Convergence Theorem.

Proposition 3 (Theorem 1, (p. 387, [20])). *Let f be a Lebesgue measurable function defined on a closed, bounded interval [a,b] with its L^p norm for $1 < p < \infty$ defined by*

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

and L^∞ norm defined by

$$\|f\|_\infty = \text{ ess. sup. } \{|f(x)| : a \leqq x \leqq b\}.$$

If $f_1, f_2, f_3, \cdots$ is a sequence of measurable functions and M is a positive constant so that

$$\|f_n\|_\infty \leq M, \quad n = 1, 2, \cdots$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. $x \in [a,b]$, then

$$\lim_{n\to\infty} \|f_n - f\|_p = 0$$

for any p satisfying $0 < p < \infty$.

Proposition 4 ([18]). *The Bochner space $L^2(0,T; W_0^{1,p}(0,1))$ is complete and separable if $1 < p < \infty$.*

In the following Proposition 5, $L(\Omega, \Lambda, \mu) = \{f : \Omega \to \mathbb{R} : f$ is μ-measurable, and $\int_\Omega f(t)d\mu(t) < \infty\}$ is a Lebesgue space.

Proposition 5 (Jensen's inequality, Theorem 1.1, (p. 221, [21])). *Let* (Ω, Λ, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\phi : I \to \mathbb{R}$ *be a convex function defined in an open interval I in* $\mathbb{R}$. *If* $f : \Omega \to I$ *is such that* $f, \phi \circ f \in L(\Omega, \Lambda, \mu)$, *then*

$$\phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu.$$

Proposition 6 (Theorem 9.16, (p. 174, [22])). *If a sequence* $\{f_n\}_{n=1^\infty}$ *of continuous functions* $f_n : A \to \mathbb{R}$ *converges uniformly on* $A \subset \mathbb{R}$ *to* $f : A \to \mathbb{R}$, *then* f *is continuous on* A.

In the following Proposition 7, $I = (a, b)$ is an open interval, possibly unbounded, and $1 < p < \infty$.

Proposition 7 (Sobolev Embedding Theorem 8.8, (p. 212, [18])). *The injection* $W^{1,p}(I) \subset C(\bar{I})$ *is compact for all* $1 < p < \infty$.

3. Existence and Uniqueness

In this section, we prove two theorems and some auxiliary lemmas, which we invoke to establish the existence and uniqueness result for the viscous Burgers' equation of power-law non-Newtonian fluids with initial and boundary conditions.

3.1. Existence

In this subsection, we will prove the existence of a solution to the problem (2). Firstly, we choose the basis $\{e_i\}_{j \in \mathbb{N}} \in L^2(0,1)$, defined as a subset of the eigenfunction of the Laplacian for the Dirichlet problem.

$$\begin{cases} -\partial_x^2 e_j = \lambda_j e_j, \quad j \in \mathbb{N}, \\ e_j = 0, \quad x \in \{0, 1\}, \end{cases} \tag{4}$$

where

$$e_j = \sqrt{2} \sin j\pi x, \quad \lambda_j = (j\pi)^2 \tag{5}$$

for $j \in \mathbb{N}$.

Since $\{e_j\}_{j \in N}$ is an orthonormal basis in $L^2(0,1)$, then it is also an orthonormal basis in $W_0^{1,p}(0,1)$. Therefore, for $f \in L^2(0,1)$, we can write that

$$f = \sum_{j=1}^{\infty} c_j(t) e_j$$

with $c_j(t) = \left(\int_0^1 |fe_j|^2 dx\right)^{1/2} < \infty, \quad j = 1, 2, \ldots$. The series $\{c_j\}_1^\infty$ converges in $L^2(0,1)$. Hence, the approximate solution $u_n(t, x)$ can be represented as follows:

$$u_n(t, x) = \sum_{j=1}^{n} c_j(t) e_j, \tag{6}$$

$$u_n(0) = u_{0n} = \sum_{j=1}^{n} c_j(0) e_j. \tag{7}$$

Moreover, we suppose that $u_n(t, x)$ satisfies the following approximating problem:

$$\int_0^1 \partial_t u_n e_j dx + \int_0^1 u_n \partial_x u_n e_j dx + \frac{1}{Re} \int_0^1 \left(|\partial_x u|^{p-2} \partial_x u\right) \partial_x e_j dx = 0, \tag{8}$$

$$u_n(0) = u_{0n}$$

for $j = 1, \dots, n$ and $t \in [0, T]$.

Remark 1. *The coefficients $c_j(0)$ can be chosen so that $\lim_{n\to\infty} u_{0n} = u(0)$ in $W_0^{1,p}(0,1)$.*

Furthermore, we prove two lemmas concerning the solution of the approximation of Problem (8). In Lemma 1, we derive an a priori estimate to control the norms $\|u_n\|_{L^2(0,1)}$ and $\|\partial_x u_n\|_{L^p(0,1)}$, whenever $1 < p < 2$. Lemma 2 establishes the existence of the solutions to the approximation of Problem (8) in $L^2(0, T; W_0^{1,p}(0,1))$.

Lemma 1. *There exists a positive constant $C_0 > 0$ that is independent of n, which satisfies the following inequality:*

$$\|u_n\|^2_{L^2(0,1)} + \frac{2}{Re}\int_0^t \|\partial_x u_n\|^p_{L^p(0,1)} d\tau \leqslant C_0$$

for all $t \in [0, T]$.

Proof. Taking $e_j := u_n$ and substituting it into Problem (8), we obtain the following:

$$\int_0^1 u_n \partial_t u_n dx + \int_0^1 u_n^2 \partial_x u_n dx + \frac{1}{Re}\int_0^1 |\partial_x u_n|^p dx = 0. \tag{9}$$

The first term of (9) can be written as follows:

$$\int_0^1 u_n \partial_t u_n dx = \frac{1}{2}\frac{d}{dt}\int_0^1 u_n^2 dx = \frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(0,1)}. \tag{10}$$

Computing the second integral of (9) by the initial condition in (8) leads us to

$$\int_0^1 u_n^2 \partial_x u_n dx = \int_0^1 \partial_x (u_n^3) dx = u_n^3\Big|_0^1 = 0. \tag{11}$$

By substituting the integrals (10) and (11) into (9) and then integrating Equation (9) from 0 to t with respect to τ, we have the following:

$$\|u_n\|^2_{L^2(0,1)} + \frac{2}{\mathrm{Re}}\int_0^t \|\partial_x u_n\|^p_{L^p(0,1)} d\tau = \|u_{n0}\|^2_{L^2(0,1)}. \tag{12}$$

The application of inequality (3) to the left-hand side of (12) leads us to the assertion of the lemma. The proof of the lemma is complete. □

Thus, we establish that u_n satisfies the stability condition in the Sobolev space, ensuring solution convergence.

Lemma 2. *There exists at least one solution of the approximating problem (8) so that $u_n \in L^2(0, T; W_0^{1,p}(0,1))$.*

Proof. It is obvious that

$$\int_0^1 e_i e_j dx = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{13}$$

We note that

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial u_n}{\partial x}\right|^{m-2}\frac{\partial u_n}{\partial x}\right) = \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2} + (m-1)\left|\frac{\partial u_n}{\partial x}\right|^{m-2}\frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial x}$$
$$= m\left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2}. \tag{14}$$

Now, we represent the approximate Equation (8) in the following form:

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = -\int_0^1 u_n \partial_x u_n e_j dx - \frac{1}{Re}\int_0^1 \frac{\partial}{\partial x}\left(\left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial u_n}{\partial x}\right) e_j dx \tag{15}$$

for $j = 1, \ldots, n$. The substitution of (14) into (15) gives the following:

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = -\int_0^1 u_n \partial_x u_n e_j dx - \frac{m}{Re}\int_0^1 \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial u_n}{\partial x} e_j dx \tag{16}$$

for $j = 1, \ldots, n$. We find the derivative of (6) with respect to t; then, by (13), the integral on the left-hand side of (16) can be written as below:

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = \int_0^1 \sum_{i=1}^n \dot{c}_i(t) e_i e_j dx = \dot{c}_j(t), \quad j = 1, \ldots, n. \tag{17}$$

For abbreviation in the notation, and denoting $\partial_x e_i$ as e_i', the first integral on the right-hand side of (16) can be written as follows:

$$\int_0^1 u_n \frac{\partial u_n}{\partial_x} e_j dx = \int_0^1 \left(\sum_{i=1}^n c_i(t) e_i\right)\left(\sum_{i=1}^n c_i(t) e_i'\right) e_j dx$$
$$= \int_0^1 \sum_{i=1}^n c_i(t) e_i' c_j(t) dx$$
$$= \left[\sum_{i=1}^n c_i(t) \int_0^1 e_i' dx\right] c_j(t) \tag{18}$$

for $j = 1, \ldots, n$. Here, $\sum_{i=1}^n c_i(t) \int_0^1 e_i' dx$ is bounded on $[0, T]$, since

$$\int_0^1 e_i' dx = a_i, \quad i = 1, \ldots, n \tag{19}$$

with $a_i < \infty$ constant numbers (see (5)). Equation (4) implies that

$$\frac{\partial^2 u_n}{\partial x^2} = \sum_{i=1}^n \lambda_i c_i(t) e_i. \tag{20}$$

Using Equation (20), we transform the second integral on the right-hand side of (16) in the following way:

$$\frac{m}{Re}\int_0^1 \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2} e_j dx = \frac{m\lambda_j c_j(t)}{Re}\int_0^1 \left|\sum_{i=1}^n c_i(t) e_i'\right|^{m-1} dx \tag{21}$$

for $j = 1, \ldots, n$.

Thus, substitutions of (17), (18), and (21) into (16) lead us to the following homogeneous ordinary system of equations with respect to t:

$$\dot{c}_j(t) = \left\{ -\sum_{i=1}^{n} c_i(t) \int_0^1 e_i' dx - \frac{m\lambda_j}{Re} \int_0^1 \left| \sum_{i=1}^{n} c_i(t) e_i' \right|^{m-1} dx \right\} c_j(t) \tag{22}$$

for $j = 1, \ldots, n$. Furthermore, we introduce the following notations:

$$\vec{c}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

with a_i $(i = 1, \ldots, n)$ from (19). Therefore, by using our notations, the homogeneous system of ODE (22) can be written as follows:

$$\dot{\vec{c}} = -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \vec{\lambda} \odot \vec{c}\, dx. \tag{23}$$

Here, $\lambda \odot \vec{c}$ is the Hadamard product of two vectors, specified by the following formula:

$$\vec{\lambda} \odot \vec{c} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \odot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_n c_n \end{pmatrix}.$$

Now, we denote the right-hand side of (23) by $\vec{F}$:

$$\vec{F}(\vec{c}(t)) = -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \vec{\lambda} \odot \vec{c}\, dx, \tag{24}$$

where

$$\begin{aligned} F_1(c_1(t)) &= -(\vec{a} \cdot \vec{c})c_1 - \tfrac{m}{Re} \textstyle\int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \lambda_1 c_1\, dx, \\ F_1(c_2(t)) &= -(\vec{a} \cdot \vec{c})c_2 - \tfrac{m}{Re} \textstyle\int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \lambda_2 c_2\, dx, \\ \cdots \quad & \cdots \quad \cdots \\ F_1(c_n(t)) &= -(\vec{a} \cdot \vec{c})c_n - \tfrac{m}{Re} \textstyle\int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \lambda_n c_n\, dx \end{aligned} \tag{25}$$

where $(\vec{a} \cdot \vec{c}) = a_1 c_1 + a_2 c_2 + \cdots + a_n c_n$ and $(\vec{e}\,' \cdot \vec{c}) = e_1' c_1 + e_2' c_2 + \cdots + e_n' c_n$ are certain real numbers. The first term on the right-hand side of (24) is the linear term. Clarifying the nonlinear component of the system, we represent (24) as follows:

$$\vec{F}(\vec{c}) = \begin{cases} -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 (\vec{e}\,' \cdot \vec{c})^{m-1} (\vec{\lambda} \odot \vec{c})\, dx & \text{for} \quad \vec{e}\,' \cdot \vec{c} > 0, \\ -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 (-\vec{e}\,' \cdot \vec{c})^{m-1} (\vec{\lambda} \odot \vec{c})\, dx & \text{for} \quad \vec{e}\,' \cdot \vec{c} < 0. \end{cases}$$

This means that Equation (23) can be rewritten as the homogeneous system of equations:

$$\dot{c}_j(t) = F_j\big(c_j(t)\big), \quad j = 1, \ldots, n. \tag{26}$$

We show that

$$\nabla \vec{F}(c) = \begin{pmatrix} \frac{\partial F_1}{\partial c_1} & \frac{\partial F_1}{\partial c_2} & \cdots & \frac{\partial F_1}{\partial c_n} \\ \frac{\partial F_2}{\partial c_1} & \frac{\partial F_2}{\partial c_2} & \cdots & \frac{\partial F_2}{\partial c_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial c_1} & \frac{\partial F_n}{\partial c_2} & \cdots & \frac{\partial F_n}{\partial c_n} \end{pmatrix}$$

is bounded and continuous. If $\vec{e} \cdot \vec{c} > 0$, then

$$\frac{\partial F_i}{\partial c_j} = -\left[a_j c_i + (\vec{a} \cdot \vec{c}) \cdot \delta_{ij}\right] - \frac{m}{Re} \int_0^1 \Big[(m-1)|\vec{e}\,' \cdot \vec{c}|^{p-2} \vec{e'}_j \lambda_i c_i + |\vec{e}\,' \cdot \vec{c}|^{p-1} \lambda_i \delta_{ij}\Big] dx.$$

Notice that

$$-\left[a_j c_i + (\vec{a} \cdot \vec{c})\delta_{ij} + \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{p-1} \lambda_i \delta_{ij} dx\right]$$

is bounded and continuous in $\vec{c}$.

We only aim to prove that

$$\int_0^1 |\vec{e}\,' \cdot \vec{c}|^{p-2} \vec{e}_j\,' \lambda_i c_i dx = \lambda_i \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{p-2} \vec{e}_j\,' c_i dx$$

is bounded and continuous at one point, where $\vec{e}\,' \cdot \vec{c} = 0$. This means that each integrand $|\vec{e}\,' \cdot \vec{c}|^{p-2} |\lambda_i e_i' c_j|$ of the integrals

$$\int_0^1 |\vec{e}\,' \vec{c}|^{p-2} \begin{pmatrix} \lambda_1 e_1' c_1 & \cdots & \lambda_1 e_n' c_1 \\ \vdots & \ddots & \cdots \\ \lambda_n e_1' c_n & \cdots & \lambda_n e_n' c_n \end{pmatrix} dx$$

is bounded if $\|\vec{c}\| \to 0$. Using Lemma 1, we have the following:

$$\|u_n\|_{L^2}^2 = \langle u_n, u_n \rangle = \left\langle \sum_{i=1}^n c_i e_i, \sum_{i=1}^n c_i e_i \right\rangle = \sum_{i=1}^n c_i^2 = \|\vec{c}\|_{\mathbb{R}^n}^2.$$

Equation (12) implies that $\|\vec{c}\|_{\mathbb{R}^n} \leqslant \|\vec{c}(0)\|$. Hence, $\vec{c}$ is bounded in $\mathbb{R}^n$. Then, there exists a sequence $\{\vec{c}_k(t)\}_1^\infty$ that converges to $\vec{c}(t)$ in $\mathbb{R}^n$ for $0 < t < \infty$. Moreover, this sequence $\{\dot{\vec{c}}_k(t)\}_1^\infty$ satisfies the following estimates:

$$\|\dot{\vec{c}}_k\| \leq \|(\vec{a} \cdot \vec{c})\vec{c}\| + \left\| \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \vec{\lambda} \odot \vec{c}\, dx \right\| \leqslant \infty.$$

Consequently, there exist functions $\vec{c}(t)$, $\dot{\vec{c}}(t)$ so that $\lim_{n\to\infty} \vec{c}_k(t) = \vec{c}(t)$ and $\lim_{n\to\infty} \dot{\vec{c}}_k(t) = \dot{\vec{c}}(t)$. Functions $\vec{c}_k(t)$, $\dot{\vec{c}}_k(t)$ are continuous and uniformly converge to $\vec{c}(t)$, $\dot{\vec{c}}(t)$ in $\mathbb{R}^n$ for all $0 \leqslant t < +\infty$. Then, using Proposition 6 on the continuity of the sum of uniformly convergent series, the functions $\vec{c}(t)$, $\dot{\vec{c}}(t)$ in $\mathbb{R}^n$ are also continuous in $\mathbb{R}^n$ for all $0 \leqslant t < +\infty$. Thus, $\vec{c}(t) \in C([0, +\infty); \mathbb{R}^n)$ and $\dot{\vec{c}}(t) \in C^\infty([0, +\infty); \mathbb{R}^n)$.

Therefore, a system of Equation (26) satisfies the condition of the existence of homogeneous system of ODE. The right-hand side of Equation (26) $\vec{F}_j(c_j(t))$ is continuous for all $j = 1, \ldots, n$ on bounded domain $[0, T]$. Consequently, there exists a unique solution

$$\vec{c}(t) = (c_1(t), c_2(t), \ldots, c_n(t))$$

of Equation (26) on $[0, T]$ that satisfies the initial condition of $\vec{c} = \vec{c}(0)$ or $\vec{c}(0) = (c_1(0), c_2(0), \ldots, c_n(0))$ for $t = 0$. The proof of the lemma is complete. □

We are now ready to complete the proof of the existence of a solution to Theorem 1.

Proof. Based on Lemma 2, there exists a weak solution u_n of the approximating problem (8) in $L^2(0, T; W_0^{1,p}(0,1))$. Moreover, based on Lemma 1, the sequence of functions $\{u_n\}_{n=1}^\infty$ and $\{\partial_t u_n\}_{n=1}^\infty$ are bounded in $L^2(0, T; W_0^{1,p}(0,1))$. Therefore, based on Proposition 4, the sequences

$\{u_n\}_{n=1}^{\infty}$ and $\{\partial_t u_n\}_{n=1}^{\infty}$ have subsequences that converge to some $u, u_t \in L^2(0,T;W_0^{1,p}(0,1))$. Denoting these subsequences, still using $\{u_n\}_{n=1}^{\infty}$ and $\{\partial_t u_n\}_{n=1}^{\infty}$, we obtain the following:

$$\lim_{n\to\infty}\int_0^T \|u_n(t)-u(t)\|_{1,p}dt = 0,$$

$$\lim_{n\to\infty}\int_0^T \|\partial_t u_n(t)-u_t(t)\|_{1,p}dt = 0,$$

which implies that

$$\lim_{n\to\infty}\|u_n(t)-u(t)\|_{L,p} = 0,$$
$$\lim_{n\to\infty}\|\partial_t u_n(t)-u_t(t)\|_{L,p} = 0$$

for a.e. t. Since $L^2(0,T;W_0^{1,p}(0,1))$ is a complete space, any sequences $\{u_n\}_{n=1}^{\infty}$, $\{\partial_t u_n\}_{n=1}^{\infty}$ from $L^2(0,T;W_0^{1,p}(0,1))$ have limits so that $u, u_t \in L^2(0,T;W_0^{1,p}(0,1))$. Since $\{u_n\}_{n=1}^{\infty} \in L^2(0,T;W_0^{1,p}(0,1))$, $\{\partial_x u_n\}_{n=1}^{\infty} \in L^2(0,T;L^p(0,1))$. Therefore, $\lim_{n\to\infty}\partial_x u_n = \partial_x u$. The proof of the theorem is complete. □

3.2. Uniqueness

In this section, we determine uniqueness of Theorem 1.

Let u be the solution of (2). Based on Section 3.1, $u, u_t \in L^2(0,T;W_0^{1,p}(0,1))$. Then, based on Proposition 7, $W_0^{1,p}(0,1) \subset C[0,1]$. Therefore,

$$u, u_t \in L^2(0,T;C^1[0,1])$$

which implies that there exists a positive number C_1 so that $C_1 = \max_{x\in[0,1]}\{|\partial_x u|, |\partial_x u_t|\}$ and $u, u_t \in L^2(0,T;C^1[0,1])$. Therefore, we have Lemma 3:

Lemma 3. *Let u be the solution of (2). Then $u, u_t \in L^2(0,T;C^1[0,1])$.*

We now prove the uniqueness of the solution to (2) by applying Jensen's, Poincare's, and Gronwall's inequalities and using the above bound C_1.

Proof. Suppose that there exist two solutions of (2): $u_1(x,t)$ and $u_2(x,t)$. Let $\hat{u} = u_1 - u_2$, and take $w := \hat{u}$ in (2). We then obtain the following:

$$\int_0^1 \hat{u}\partial_t\hat{u}dx + \int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx + \frac{1}{\mathrm{Re}}\int_0^1 \left(|\partial_x u_1|^{p-2}\partial_x u_1 - \left|\partial_x u_2\right|^{p-2}\partial_x u_2\right)\hat{u}dx = 0 \tag{27}$$

with $1 < p < 2$. The first integral on the left-hand side of (27) can be written as (10)

$$\int_0^1 \hat{u}\partial_t\hat{u}dx = \frac{1}{2}\frac{d}{dt}\|\hat{u}\|_{L^2(0,1)}. \tag{28}$$

We note that

$$(u_1-u_2)(\partial_x u_1 - \partial_x u_2) = u_1\partial_x u_1 - u_2\partial_x u_1 - u_1\partial_x u_2 + u_2\partial_x u_2.$$

Therefore,

$$u_1\partial_x u_1 - u_2\partial_x u_2 = (u_1 - u_2)(\partial_x u_1 - \partial_x u_2) + u_2\partial_x(u_1 - u_2) + (u_1 - u_2)\partial_x u_2 = \hat{u}\partial_x\hat{u} + u_2\partial_x\hat{u} + \hat{u}\partial_x u_2.$$

Hence, the second integral of (27) can be represented in the following way:

$$\int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx = \int_0^1 \hat{u}^2\partial_x\hat{u}dx + \int_0^1 u_2\hat{u}\partial_x\hat{u}dx + \int_0^1 \hat{u}^2\partial_x u_2 dx. \quad (29)$$

Based on (11), the first term on the right-hand side of Equation (29) equals zero. By invoking Lemma 3, we have the following bound to the third integral on the right-hand side of (29):

$$|\int_0^1 \hat{u}^2\partial_x u_2 dx| \leqslant \max_{x\in[0,1]}|\partial_x u_2| \int_0^1 \hat{u}^2 dx = C_1\|\hat{u}\|^2_{L^2(0,1)}, \quad (30)$$

where $C_1 = \max_{x\in[0,1]}|\partial_x u_2|$. Integrating by parts the second term on the right-hand side of Equation (29), we have the same estimate as (30):

$$\left|\int_0^1 u_2\hat{u}\partial_x\hat{u}dx\right| = \left|-\frac{1}{2}\int_0^1 \hat{u}^2\partial_x u_2 dx\right| \leqslant \frac{C_1}{2}\|\hat{u}\|^2_{L^2(0,1)}.$$

This means that we obtain the following estimate:

$$\int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx \leqslant \frac{3}{2}C_1\|\hat{u}\|^2_{L^2(0,1)} \quad (31)$$

for the second integral of (27). Furthermore, we bound the third term on the left-hand side of (27). Therefore, we apply the following inequality [23]:

$$(p-1)|b-a|^2 \leq \langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle$$

for $b - a > 0$, $a, b \in \mathbb{R}^n$ and $1 < p < 2$ to the last term of the Equation (27). We then have the following:

$$\int_0^1 |\partial_x\hat{u}|^2\hat{u}_x dx \leqslant \int_0^1 \left(|\partial_x u_1|^{p-2}\partial_x u_1 - |\partial_x u_2|^{p-2}\partial_x u_2\right)\hat{u}_x dx. \quad (32)$$

The left-hand side of (32) can be written as follows:

$$\int_0^1 |\partial_x\hat{u}|^2\partial_x\hat{u}dx = \int_0^1 |\partial_x\hat{u}|^3 dx = \|\partial_x u\|^3_{L^3(0,1)}.$$

Based on Jensen's inequality, we bound the norm in the last equation from below:

$$\|\partial_x u\|_{L^2(0,1)} = \left(\int_0^1 |\partial_x\hat{u}|^2 dx\right)^{\frac{1}{2}} = \left[\left(\int_0^1 |\partial_x\hat{u}|^{3\cdot\frac{2}{3}} dx\right)^{\frac{2}{3}\cdot\frac{1}{2}}\right]^{\frac{3}{2}} \leqslant \left(\int_0^1 |\partial_x\hat{u}|^{3\cdot\frac{2}{3}\cdot\frac{3}{2}} dx\right)^{\frac{1}{3}} \leqslant \|\partial_x u\|_{L^3(0,1)}.$$

The last inequality and the current inequality (32) imply that

$$\|\partial_x u\|^3_{L^2(0,1)} \leqslant \int_0^1 \left(|\partial_x u_1|^{p-2}\partial_x u_1 - |\partial_x u_2|^{p-2}\partial_x u_2\right)\hat{u}_x dx. \quad (33)$$

By combining inequalities (33), (31), and (28), Equation (27) implies that

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} + \frac{2}{Re}\|\partial_x \hat{u}\|^3_{L^2(0,1)} \leqslant 3C_1\|\hat{u}\|^2_{L^2(0,1)}.$$

Exploiting Poincare's inequality (3) to the last inequality leads us to the following:

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} + \frac{2}{\mathrm{Re}}\|\hat{u}\|^3_{L^2(0,1)} \leqslant 3C_1\|\hat{u}\|^2_{L^2(0,1)}.$$

Combining the similar terms in the last inequality, we conclude that there exists a non-negative constant K so that

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} \leqslant K\|\hat{u}\|^2_{L^2(0,1)} \tag{34}$$

with $K = \left(3C_1 + \frac{2}{Re}\|\hat{u}\|_{L^2(0,1)}\right)$. Invoking Gronwall's inequality (see Proposition 1) to (34), we find that $\hat{u} = 0$. Thus, the solution of Problem (9) is unique. Now, the proof of Theorem 1 is complete. □

4. Numerical Testing

In this section, we first create a data set by solving the Reynolds number numerically with the prescribed power-law rheology parameters. The corresponding parameters and values for the Reynolds number are listed in Table 1. Problem (1) was solved using the finite element software COMSOL Multiphysics 6.0. We chose COMSOL Multiphysics for this study because of its robust finite element capabilities, which make it well suited for handling nonlinear PDEs, complex boundary conditions, and adaptive meshing. Unlike the finite difference and spectral methods, which have limitations with irregular domains and strong gradients or custom FEM implementations that require extensive coding, COMSOL offers optimized solvers and a reliable framework for accurately solving this class of equations. To use the COMSOL software, the equation in (1) was written in a standard form for the solver:

$$e_a\frac{\partial^2 u}{\partial t^2} + d_a\frac{\partial u}{\partial t} + \nabla\cdot(-c\nabla u - \alpha u + \gamma) + \beta\cdot\nabla u + au = f,$$

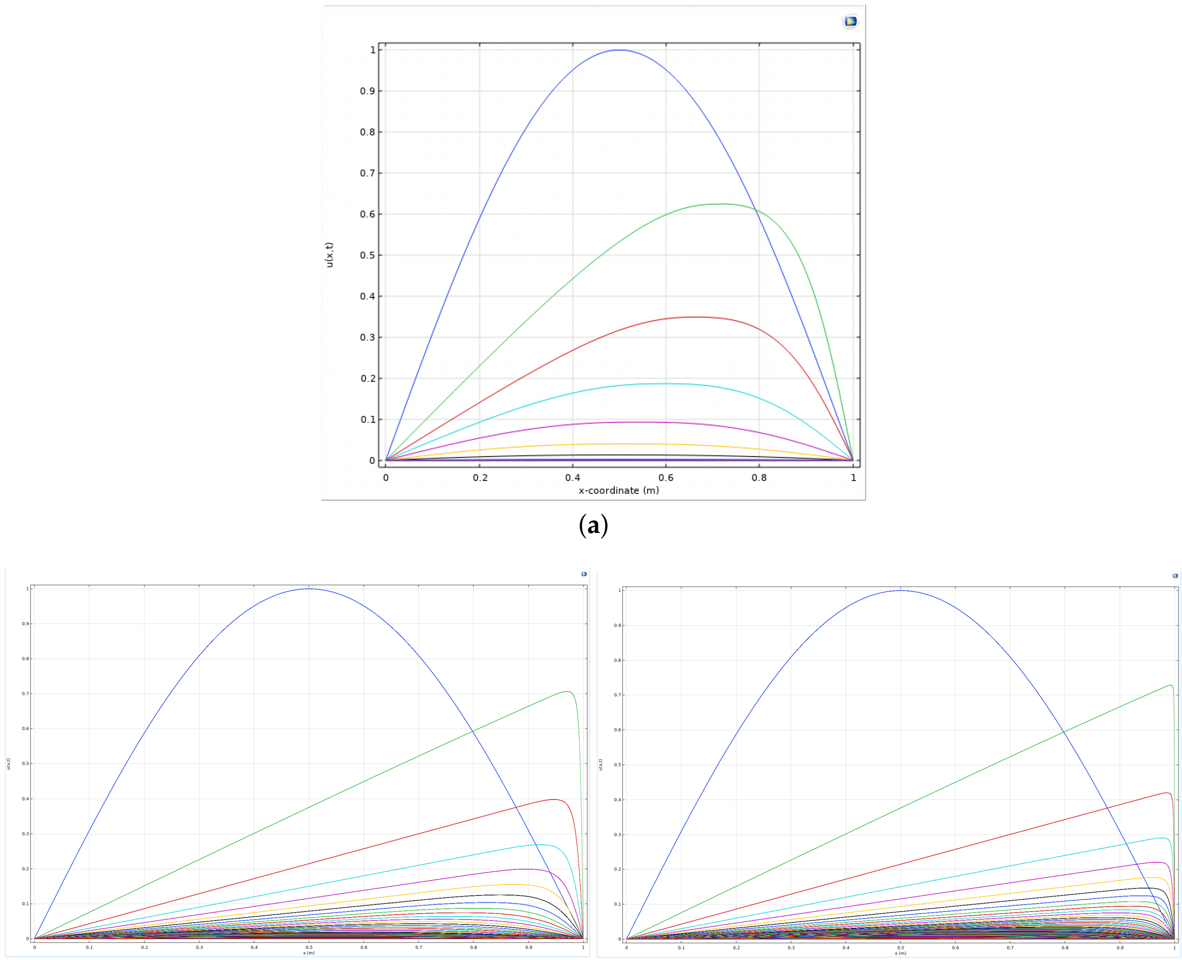
where $\nabla = \partial_x$ and $u = u(t,x)$. We constructed the numerical solutions of Problem (1), using the coefficients $e_a = 0$, $d_a = 1$, $a = 0$, $f = 0$, $\alpha = 0$, $\gamma = 0$, $\beta = u$, and $c = \frac{1}{Re}(|\nabla u|)^{m-1}$ in the last equation. The Dirichlet boundary condition was applied on both ends of the computational domain.

Table 1. Parameter values for power-law fluids.

Parameters	Case 1	Case 2	Case 3
ρ (kg/m^3)	1000	1200	1100
u (m/s)	0.10	2.0	10.00
L (m)	0.05	0.10	0.05
n	0.70	0.80	0.70
K (Pa·s^n)	1.00	2.00	2.00
γ (s^{-1})	1000	500	200
Re	9.98	165.4	537.1

Figure 1 shows the simulated solutions that were obtained for Burgers' equation based on a power-law model with three different Reynolds numbers. The values ρ, u, L, n, K, γ, and Re were selected based on previous studies of non-Newtonian fluid flows, representing different viscosity gradient scenarios. In each case, the velocity profile was shown for several instances of time. It can be observed that within a finite time T, the solution u and its derivative u_x are bounded in $[0, 1]$, as predicted by our theoretical results.

Figure 1 illustrates the relationship between the Reynolds number and solution behavior. For the large values of Reynolds numbers, the maximum velocity of the power-law fluid flow, which is our solution $u(x, t)$ to (2), is located closer to 1 for x. The amplitude of the solution $||u(t)||_\infty$ converges to 0 as $t \to \infty$.



(**a**)

(**b**) (**c**)

Figure 1. The solution $u(x, t)$ for $0 \leq x \leq 1$ and a sequence of discrete times, plotted in different colors, as the velocity of the flow of the power-law fluid between two parallel plates at $x = 0$ and $x = 1$. (**a**) Reynolds number $Re = 9.98$. (**b**) Reynolds number $Re = 165.4$. (**c**) Reynolds number $Re = 537.1$.

5. Conclusions

This work proves the existence and uniqueness of weak solutions of the viscous Burgers' equation for the isothermal flow of power-law non-Newtonian fluids with initial and boundary conditions. The research extends the classical result for the Newtonian fluids with $p = 2$ to non-Newtonian fluids with a power-law index of $1 < p < 2$ in the context of Sobolev spaces. In this work, we demonstrated the existence and uniqueness of the solution for the initial boundary value problem for the viscous Burgers' equation with a p-Laplace operator in $L^2(0, T; W_0^{1,p}(0, 1))$, and we showed this in $L^2(0, T; C^1[0, 1])$. A numerical experiment on COMSOL Multiphysics was conducted to provide an illustration of the solution, which confirms the theoretical part.

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