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Generalized uncertainty principle corrections in Rastall–Rainbow Casimir wormholes

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Abstract We explore wormhole solutions sourced by Casimir energy density involving generalized uncertainty principle corrections within the framework of Rastall–Rainbow gravity. The questions of traversability and stability, as well as the presence of exotic matter, are carefully investigated. In particular, the stability issue is addressed via an approach that has not been previously employed in the context of wormholes. This method, which represents an improved version of the so-called Herrera cracking technique, has the potential to yield novel insights in the field of wormhole geometries.

1 Introduction

Wormholes (WHs) are fascinating objects predicted by general relativity (GR) that manifest as tunnel-like structures connecting two distinct spacetime regions. Spurred by the possibility discussed in the seminal paper by Morris and Thorne of enabling human interstellar travel [1], WH solutions are now widely studied in the literature [2–4]. Two key aspects define the WH structures. The first distinctive feature is the *throat*, a minimal surface area whose characteristics are crucial for understanding the WH physics and stability. Additionally, the presence of matter fields that violate the null energy condition (NEC), referred to as *exotic matter*,

which is essential for keeping WH mouths open and hence ensuring the *traversability*. This requirement is linked to the well-known *flaring-outward* condition, which imposes precise constraints on the properties of the matter sourcing the WH.

Various WH models have been devised in the literature attempting to avoid or, at least, reduce the amount of exotic matter required to support the WH, or to find novel and alternative explanations for its origin. For instance, in thin-shell WHs [5–11] exotic matter is concentrated within a surface layer localized at the throat, while phantom WHs [12–15] take into account a phantom energy equation of state (i.e., $w < -1$, w being the ratio between the pressure and the energy density) to construct and sustain a traversable WH. This latter framework has been further expanded to include WHs endowed with generalized Chaplygin gas [16–18] and polytropic phantom energy [19–21]. Moreover, recent literature has explored novel categories such as replica WHs [22–25], which are supposed to play a significant role in the black hole information problem within the context of Euclidean quantum gravity, braneworld WHs [26–28], which borrow ideas and techniques from brane cosmology, and an intriguing kind of WHs constructed via the so-called black bounces spacetimes [29–31], which broaden the class of regular black holes [32].

A natural scenario for the existence of WHs is provided by extended or alternative gravity paradigms [33–35]. These include a plethora of WH candidates, ranging from $f(R)$ [36–39], $f(T)$ [40], $f(Q)$ [41–43], and $f(R, T)$ [44–46] gravity models, to hybrid metric-Palatini gravity [47–

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50], Einstein–Gauss–Bonnet [51–56], scalar-tensor [57–59], Einstein–Cartan theories [60–62], non-local gravity [63] and higher dimensions [64,65]. A recently proposed framework, that is attracting the scientific community, is the so-called Rastall–Rainbow (RR) gravity [66], which combines Rastall [67] and Rainbow [68] theories. The key idea of the first scenario consists in introducing a quantity λ, dubbed the Rastall parameter, which acts as a coupling constant amending the ordinary stress-energy tensor conservation law $T^\nu_{\mu;\nu} = 0$ via the introduction of a term proportional to the derivative of the Ricci scalar. The Rainbow model calls for two arbitrary functions, usually denoted by Ξ and Σ, which depend on the energy of test particles moving through the spacetime and modify the standard relativistic energy–momentum dispersion relation $E^2 - p^2 = m^2$ (with E, p, and m being the energy, momentum and rest mass of the particle, respectively). By combining the principles of these two models, one finds that the Einstein field equations assume a generalized form in the RR pattern, where λ embodies the coupling between geometry and matter fields, and the underlying spacetime properties are influenced by energy owing to the contributions of Ξ and Σ. In this context, WH solutions have been discovered, proving that their occurrence is allowed for specific combination of the free RR parameters and matter equations of state that mitigate the NEC breach [69–71].

Considering that NEC is naturally violated in quantum field theory, a new prototype called Casimir WH has been put forward in Ref. [72] (see also Refs. [73,74]). This solution exploits the very well-known fact that the negative Casimir energy density can represent a potential source of exotic matter realizable in a laboratory, thereby enabling the generation of traversable WHs. Shortly after this proposal, several extensions were conceived. Among them, we mention Casimir WHs that involve the corrections to the Casimir effect arising from the generalized uncertainty principle (GUP) [75–80]. The GUP is a pivotal component of quantum gravity models inherently equipped with a minimal length factor β, which revises the Heisenberg uncertainty principle to account for quantum gravitational effects at small scales [81–83]. These contributions can significantly affect both the traversability and stability of Casimir WHs, thus offering valuable hints into how quantum-gravity regimes can influence WH geometries.

Driven by these considerations, in this paper we deal with a novel class of static and spherically symmetric Casimir WHs incorporating GUP adjustments within the framework of RR gravity. The role of β in the fundamental questions of traversability, stability, and the presence of exotic matter is carefully assessed. In particular, the WH stability is evaluated by adopting a method known as *Herrera cracking technique*, which, to the best of our knowledge, has never been employed in its most recent and revised form in WH research settings.

The paper is organised as follows. After outlining the main features of RR gravity in Sect. 2, in Sect. 3 we describe the Casimir effect featuring GUP modifications. GUP-corrected Casimir WHs in RR gravity are then examined in Sect. 4. Discussion and final remarks are reported in Sect. 5.

2 The Rastall–Rainbow gravity

The RR gravity originates from the fusion of the Rastall and Rainbow theories. We begin the section by first outlining the Rastall and Rainbow frameworks separately (see Sects. 2.1 and 2.2). Subsequently, we delve into the combined RR approach in Sect. 2.3.

2.1 The Rastall gravity

The Rastall gravity [67] is a phenomenological extension of GR featuring a modified conservation law of the energy–momentum tensor, where the covariant derivative of $T_{\mu\nu}$ does not vanish, but, instead, it is related to the Ricci scalar R via an undetermined constant $\bar\lambda$. Explicitly, this revised relation takes the form

$$T^\nu_{\mu;\nu} = \bar\lambda R_{;\mu}, \tag{1}$$

and entails the following reformulation of the field equations:

$$R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = 8\pi G \left(T^\nu_\mu - \bar\lambda \delta^\nu_\mu R\right), \tag{2}$$

which can also be expressed as

$$R^\nu_\mu - \frac{\lambda}{2}\delta^\nu_\mu R = 8\pi G T^\nu_\mu. \tag{3}$$

Here, we have defined $\bar\lambda := \frac{1-\lambda}{16\pi G}$, λ representing the Rastall free parameter that incorporates the nonminimal curvature-matter coupling. The standard Einstein equations are thus recovered when $\lambda = 1$ (or $\bar\lambda = 0$).

2.2 The Rainbow gravity

The Rainbow gravity [68] stems from the extension to curved spacetimes of the deformed (or doubly) special relativity introduced in Ref. [84]. A crucial aspect of this theory is the modification of the standard relativistic dispersion relation, which becomes

$$E^2 \Xi^2(x) - p^2 \Sigma^2(x) = m^2, \tag{4}$$

where $x := E/E_p$ is a dimensionless ratio between the energy E of a probe particle and the Planck energy E_p, while the correction terms $\Xi(x)$ and $\Sigma(x)$ are known as rainbow functions. In the infrared limit, where x approaches zero, they satisfy the conditions

$$\lim_{x \to 0} \Xi(x) = 1, \quad \lim_{x \to 0} \Sigma(x) = 1, \tag{5}$$

thus restoring the standard energy–momentum dispersion relation.

In the Rainbow theory, the spacetime metric depends on the energy [68] and hence the field equations assume the generalized form

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = 8\pi G(x) T_{\mu\nu}(x), \tag{6}$$

where $G(x)$ is an energy-dependent coupling function usually rescaled as $G(x) = g^2(x) G$, G being the Newton constant.

2.3 The Rastall–Rainbow gravity

The RR gravity combines the concepts of Rastall and Rainbow theories [66]. In this unified formalism, the modified Einstein equations can be constructed starting from Eq. (3) upon considering an energy-dependent metric $g_{\mu\nu}(x)$ and coupling constant $G(x)$. Therefore, the RR field equations read as

$$R_{\mu\nu}(x) - \frac{\lambda}{2} g_{\mu\nu}(x) R(x) = k(x) T_{\mu\nu}(x), \tag{7}$$

where λ is supposed to be independent of the energy and $k(x) := 8\pi G(x)$. By adding and subtracting the term $(1/2) g_{\mu\nu} R$, the above equation can be written in a more compact way as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \tau_{\mu\nu}, \tag{8}$$

which resembles the ordinary Einstein equations with

$$\tau_{\mu\nu} := T_{\mu\nu} - \frac{(1-\lambda)}{2(1-2\lambda)} g_{\mu\nu} T, \tag{9}$$

denoting an effective energy–momentum tensor.

In the next section, we make a small digression on static and spherically symmetric WH solutions in RR gravity. These will be the starting point of our subsequent analysis. Hereafter, we will assume $G(x) = 1 = G$ for simplicity; see also Refs. [66,85].

2.3.1 Wormhole solutions

The most general line element for a static and spherically symmetric geometry in RR gravity can be expressed as

$$ds^2 = -\frac{e^{2\Phi(r)}}{\Xi^2(x)} dt^2 + \frac{dr^2}{\Sigma^2(x)\left(1 - \frac{b(r)}{r}\right)} + \frac{r^2}{\Sigma^2(x)} (d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

where we have employed standard spherical coordinates (t, r, θ, ϕ). This metric represents a WH spacetime with $\Phi(r)$ and $b(r)$ denoting the *redshift* and *shape* functions, respectively,

provided certain well-established conditions are met [1]. First of all, a throat connecting two asymptotically flat regions of the spacetime has to exist. The throat corresponds to a minimal surface area and satisfies the condition $b(r_0) = r_0$. Moreover, the redshift function $\Phi(r)$ has to remain finite throughout the entire spacetime to prevent the presence of horizons. Last, the shape function is required to satisfy the so-called flaring-outward condition near the throat (see Sect. 4.2.1).

In order to search for analytical WH solutions, we consider an anisotropic perfect-fluid energy–momentum tensor

$$T_{\mu\nu} = (\rho + p_t) u_\mu u_\nu + p_t g_{\mu\nu} - (p_t - p_r) \chi_\mu \chi_\nu, \tag{11}$$

where u^μ represents the fluid four-velocity, χ^μ the unit radial vector satisfying $\chi_\mu \chi^\mu = 1$, $\rho = \rho(r)$ the energy density, and $p_r = p_r(r)$ and $p_t = p_t(r)$ the pressure components along the radial and transverse directions, respectively. By evaluating Eq. (8) for the WH geometry (10), one obtains the following expressions for the components of the energy–momentum tensor [69]:

$$8\pi \bar{\rho} = \frac{b'}{r^2}, \tag{12a}$$

$$8\pi \bar{p}_r = 2\left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} - \frac{b}{r^3}, \tag{12b}$$

$$8\pi \bar{p}_t = \left(1 - \frac{b}{r}\right) \left[\Phi'' + \Phi'^2 - \frac{b'r - b}{2r(r-b)} \Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r} \right], \tag{12c}$$

where the prime denotes the derivative with respect to the radial coordinate r. Here, we have defined the effective energy density $\bar{\rho}$ and the effective radial and tangential pressure components $\bar{p}_r$ and $\bar{p}_t$, respectively, as

$$\bar{\rho} := \frac{1}{\Sigma^2(x)} \left[\gamma_1 \rho + \gamma_2 p_r + 2\gamma_2 p_t \right], \tag{13a}$$

$$\bar{p}_r := \frac{1}{\Sigma^2(x)} \left[\gamma_2 \rho + \gamma_1 p_r - 2\gamma_2 p_t \right], \tag{13b}$$

$$\bar{p}_t := \frac{1}{\Sigma^2(x)} \left[\gamma_2 \rho - \gamma_2 p_r + \gamma_3 p_t \right], \tag{13c}$$

with

$$\gamma_1 := \frac{1 - 3\lambda}{2(1 - 2\lambda)}, \qquad \gamma_2 := \frac{1 - \lambda}{2(1 - 2\lambda)},$$
$$\gamma_3 := -\frac{\lambda}{1 - 2\lambda}. \tag{14}$$

It is worth noticing that Eq. (12) agrees with the conventional Tolman–Oppenheimer–Volkoff equations for a static spherically symmetric anisotropic fluid in the limit $\lambda = \Sigma(x) = 1$.

The set (12) gives rise to a coupled system of three differential equations involving five unknowns: $\Phi(r)$, $b(r)$, $\rho(r)$,

$p_r(r)$, and $p_t(r)$. Mathematically, this system is indeterminate. To overcome this issue, one typically assumes a specific form for these quantities, thus finding that WHs are typically sourced by exotic matter [1]. In this paper, we will explore traversable and stable WHs sourced by the negative Casimir energy density encompassing GUP corrections. For this reason, in the next section, we will present the Casimir effect including modifications provided by the GUP.

3 Casimir effect with generalized uncertainty principle corrections

The Casimir effect involves two parallel uncharged conductors experiencing an attractive force due to the distortion of the vacuum of the electromagnetic field, which can be related to the zero-point energy of quantum electrodynamics [86]. Its primary relevance lies in the occurrence of a negative energy density ρ_{CE} that can potentially be generated under laboratory conditions [87]. The explicit expression of ρ_{CE} reads as (henceforth, we set $\hbar = 1$)

$$\rho_{CE}(a) = -\frac{\pi^2}{720} \frac{1}{a^4}, \tag{15}$$

where a denotes the distance between the conductive plates.

Since the prediction originally made by Casimir, this phenomenon has been experimentally validated by many researchers [88–92], and currently represents a fundamental research topic which is widely studied in the literature, ranging from cosmology [93,94] to supergravity and superstring theory [95–97]. Recently, the Casimir effect has been investigated in quantum gravity patterns featuring a modification of the standard position-momentum Heisenberg uncertainty principle [98], known as GUP (see e.g. Refs. [81–83,99–106] for further details). Explicitly, the GUP can be written as

$$\Delta X \Delta P \geq \frac{1}{2} \left[1 + \beta (\Delta P)^2 \right], \tag{16}$$

where we have defined, for any operator $\hat{O}$, the variance

$$(\Delta O)^2 := \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2, \tag{17}$$

and the factor β is a deforming parameter which puts forth the existence of a minimum length scale in the model. Consequently, the standard Heisenberg commutator relation acquires a correction term [98]:

$$\left[\hat{X}, \hat{P} \right] = i \left(1 + \beta \hat{P}^2 \right). \tag{18}$$

In this framework, a crucial role is fulfilled by the so-called maximally localized quantum states $|\psi^{ML}\rangle$, which extend the usual properties of the position eigenstates $|X\rangle$. These states minimize the uncertainty ΔX and are centered around

an average position $\langle \psi^{ML} | \hat{X} | \psi^{ML} \rangle$ with the best possible resolution [107].

When considering n spatial dimensions, the generalized commutation relations (18) are modified to the form

$$\left[\hat{X}_A, \hat{P}_B \right] = i \left[f(\hat{P}^2) \delta_{AB} + g(\hat{P}^2) \hat{P}_A \hat{P}_B \right], \tag{19}$$

where the indices A and B range from 1 to n, and the functions $f(\hat{P}^2)$ and $g(\hat{P}^2)$ are not completely arbitrary as they are subject to some relations stemming from the translational and rotational invariance of Eq. (19). For the general case with $n > 1$, different models arise based on the choice of $f(\hat{P}^2)$ and $g(\hat{P}^2)$, leading to distinct constructions of the maximally localized quantum states. In the literature, two main proposals have been devised to develop these states: the approach pursued by Kempf, Mangano, and Mann (KMM) [108], and the one followed by Detournay, Gabriel, and Spindel (DGS) [109]. Due to the different properties assumed by the $|\psi^{ML}\rangle$ states in these two patterns, the standard Casimir energy density (15) receives a correction factor. This contribution can be traced back to the presence of the minimal length scale brought in by the GUP, which limits the resolution of small distances in the spacetime. Therefore, up to first-order terms in β, the modified Casimir energy density can be written as [98]

$$\rho(a) = -\frac{\pi^2}{720} \frac{1}{a^4} \left[1 + \xi_i \frac{\beta}{a^2} \right],$$

$$\text{where } \begin{cases} \xi_{KMM} = \pi^2 \left[\frac{28 + 3\sqrt{10}}{14} \right] \\ \\ \xi_{DGS} = 4\pi^2 \left[\frac{3 + \pi^2}{21} \right] \end{cases}, \tag{20}$$

depending on whether one employs the KMM or the DGS pattern.

The GUP-corrected Casimir energy density (20) represents a unique source of exotic matter that can sustain a WH. The resulting geometry, evaluated within the framework of RR gravity, will be the focus of the next section.

4 The wormhole solution

Although traversable WHs are theoretically possible in classical GR, their existence relies on the presence of exotic matter and the associated possibility of contravening the NEC [1]. On the other hand, quantum mechanics is the natural arena allowing for the NEC violation, and the Casimir effect, which inherently enables a negative energy density, represents an example. Therefore, in this section, we will deal with GUP-corrected Casimir WHs within the framework of RR gravity. For simplicity, we hereafter consider zero-tidal-force WHs, for which the redshift function $\Phi(r)$ can be con-

sidered constant [1]. We begin our investigation by analyzing the embedding diagrams in Sect. 4.1. After that, we study WH traversability and stability in Sect. 4.2, and we calculate the amount of exotic matter required to keep the WH throat open in Sect. 4.3. We conclude the section by evaluating the light deflection angle triggered by the WH gravitational field (see Sect. 4.4).

4.1 Embedding diagrams

In the hypothesis of zero-tidal-force WHs, the differential system (12) takes the much simpler form

$$b' = 8\pi r^2 \bar{\rho}, \tag{21a}$$

$$b = -8\pi r^3 \bar{p}_r, \tag{21b}$$

$$rb' - b = -16\pi r^3 \bar{p}_t. \tag{21c}$$

These equations lead to the useful identity $\bar{\rho} + \bar{p}_r = -2\bar{p}_t$, which translates into (cf. Eqs. (13) and (14))

$$p_t = -\left(\frac{\gamma_1 + 3\gamma_2}{2\gamma_3}\rho + \frac{1}{2}p_r\right), \tag{22}$$

which, in turn, allows to write, from Eq. (13a),

$$\bar{\rho} = \frac{\rho}{\Sigma^2(x)\lambda}. \tag{23}$$

In the above formula, ρ is represented by the GUP-corrected Casimir energy density (20), with the plate separation distance a replaced by the radial coordinate r. In this way, Eq. (21a) can be readily integrated and yields

$$b(r) = \frac{\pi^3 \left(\beta\xi_i + 3r^2\right)}{270\lambda r^3 \Sigma^2(x)} + \mathcal{C}, \tag{24}$$

where the integration constant $\mathcal{C}$ can be determined by imposing the condition $b(r_0) = r_0$, thus obtaining

$$b(r) = r_0 + \frac{\pi^3}{270\lambda\Sigma^2(x)}\left[3\left(\frac{1}{r} - \frac{1}{r_0}\right) + \beta\xi_i\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right)\right]. \tag{25}$$

Notice that $b(r)/r$ goes to zero at large distances, and hence the (energy-dependent) asymptotic flatness condition is fulfilled.

It follows, from Eqs. (21b) and (25), that the radial pressure is given by

$$p_r(r) = \frac{\pi^2}{2160\lambda}\left[\frac{3}{r^3}\left(\frac{\lambda - 2}{r} + \frac{1}{r_0}\right)\right.$$
$$\left. + \frac{\beta\xi_i}{r^3}\left(\frac{3\lambda - 4}{r^3} + \frac{1}{r_0^3}\right)\right] - \frac{r_0\Sigma^2(x)}{8\pi r^3}, \tag{26}$$

which, owing to Eq. (22), leads to the tangential pressure expression

$$p_t(r) = \frac{\pi^2}{4320\lambda}\left[\frac{3}{r^3}\left(\frac{2\lambda}{r} - \frac{1}{r_0}\right)\right.$$
$$\left. + \frac{2\beta\xi_i}{r^3}\left(\frac{3\lambda - 1}{r^3} - \frac{1}{2r_0^3}\right)\right] + \frac{r_0\Sigma^2(x)}{16\pi r^3}. \tag{27}$$

Equations (25)–(27), jointly with Casimir energy density formula (20) and the condition $\Phi(r) = const.$, completely characterize the WH geometry.

The embedding diagrams (see Fig. 1) can be traced following the standard procedure outlined in Ref. [1]. First of all, we take advantage of the symmetries and consider the equatorial time-fixed two-dimensional slice (cf. Eq. (10))

$$ds^2 = \frac{1}{\Sigma^2(x)\left(1 - \frac{b(r)}{r}\right)}dr^2 + \frac{r^2}{\Sigma^2(x)}d\phi^2. \tag{28}$$

After that, we express the line element of the three-dimensional Euclidean embedding space via cylindrical coordinates (r, z, ϕ) as

$$ds^2 = dr^2 + dz^2 + r^2 d\phi^2. \tag{29}$$

Then, the axially symmetric embedded surface admits the metric

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right]dr^2 + r^2 d\phi^2, \tag{30}$$

where $z(r)$ satisfies the following equation, obtained upon comparing Eqs. (28) and (30):

$$\frac{dz}{dr} = \pm\left[\frac{1}{\Sigma^2(x)}\left(\frac{1}{1 - \frac{b(r)}{r}} - 1\right)\right]^{\frac{1}{2}}. \tag{31}$$

With these considerations in mind, let us discuss the physically consistent WH solutions in the next section.

4.2 Traversability and stability

In this section, we explore the interplay between quantum-gravity effects and WH geometry by investigating how the GUP parameter β influences the traversability and stability of the WH. We assess the first aspect in Sect. 4.2.1, and then deal with the second by resorting to two different criteria in Sects. 4.2.2 and 4.2.3. We conclude the section with a brief
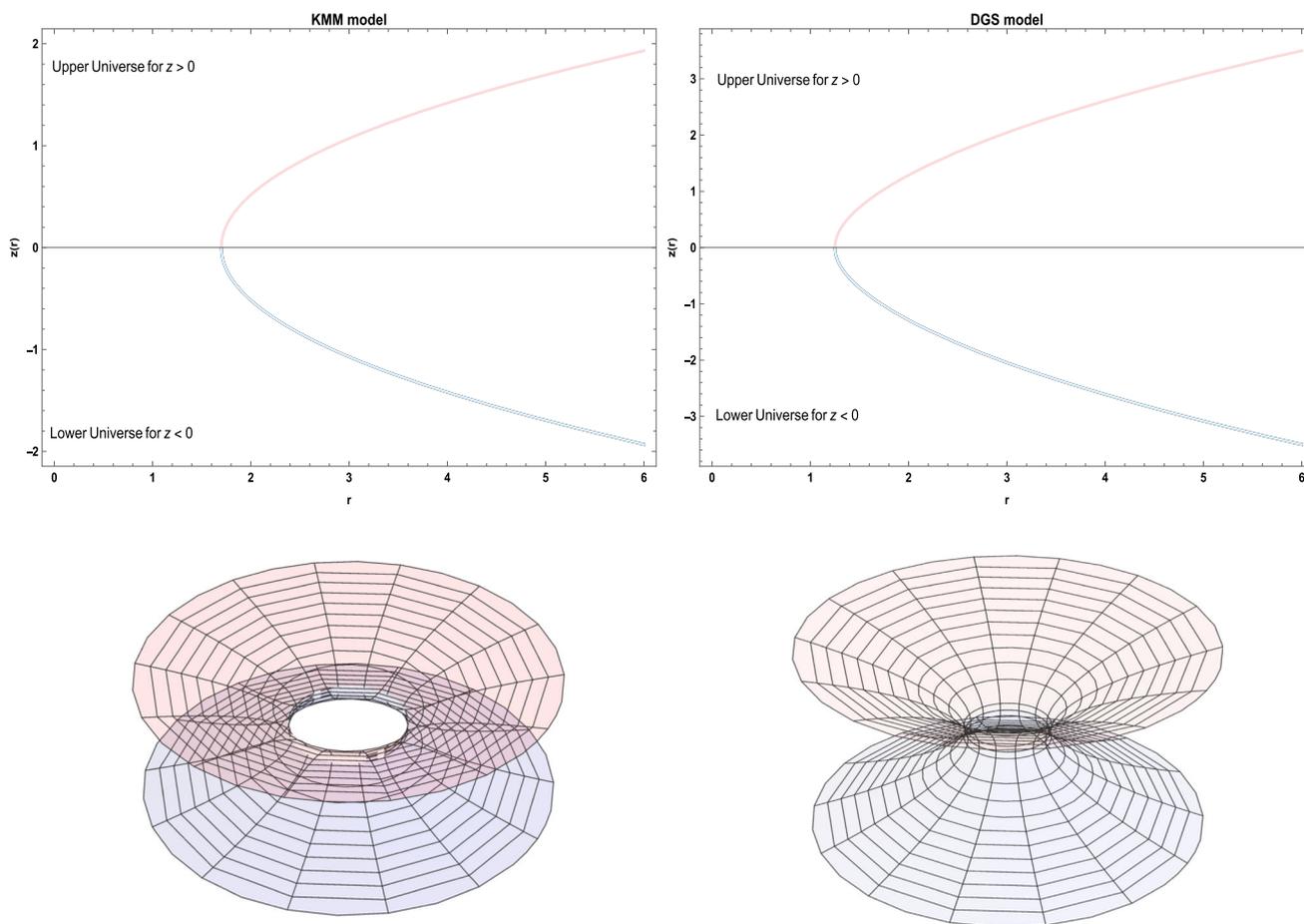
Fig. 1 Embedded surface $z(r)$ representing the solution of Eq. (31) (top panel) and embedding diagram (bottom panel) for the GUP-corrected Casimir WH for both the KMM and DGS models. We have chosen the following values: $\lambda = 0.34$, $\Sigma(x) = 1.3$, $\beta = 0.17$, while $r_0 = 1.7$ (KMM model) or $r_0 = 1.25$ (DGS model)

analysis of Casimir WHs comprising no GUP corrections, which admit $\beta = 0$ (see Sect. 4.2.4).

4.2.1 Flaring-outward condition

The WH traversability is related to the fulfillment, near the throat, of the well-known flaring-outward condition [1]

$$b(r) - rb'(r) > 0, \tag{32}$$

which boils down to

$$b'(r_0) < 1, \tag{33}$$

at $r = r_0$. Starting from Eq. (25), we find

$$b(r) - rb'(r) = r_0 + \frac{\pi^3}{270\lambda\Sigma^2}$$

$$\times \left[3\left(\frac{2}{r} - \frac{1}{r_0}\right) + \beta\xi_i\left(\frac{4}{r^3} - \frac{1}{r_0^3}\right) \right], \tag{34}$$

which readily gives at the throat

$$b'(r_0) = -\frac{\pi^3\left(1 + \beta\xi_i/r_0^2\right)}{90\lambda\Sigma^2 r_0^2}. \tag{35}$$

The last relation agrees with the upper bound (33) provided

$$\lambda > 0, \tag{36}$$

a hypothesis that we henceforth adopt. Now, bearing in mind the formula (34) and upon rescaling the radial variable according to

$$r = \alpha r_0, \qquad \alpha \geq 0, \tag{37}$$

the flaring-outward condition (32) is equivalent to requiring that

$$\mathcal{F}(\alpha) > 0, \tag{38}$$

where the real-valued function $\mathcal{F}(\alpha)$ is defined as

$$\mathcal{F}(\alpha) := A\alpha^3 + B\alpha^2 + C, \tag{39}$$

the coefficients A, B, and C being given by

$$A = 270\lambda \Sigma^2 r_0^2 - 3\pi^3 - \pi^3 \beta \xi_i / r_0^2, \tag{40a}$$
$$B = 6\pi^3, \tag{40b}$$
$$C = 4\pi^3 \beta \xi_i / r_0^2. \tag{40c}$$

A qualitative analysis of the $\mathcal{F}$ function can be given as follows. First of all, from the Descartes rule of signs we find that the algebraic cubic equation

$$\mathcal{F}(\alpha) = 0, \tag{41}$$

has one real root, say α_1, and two complex conjugate solutions. In particular, if $A > 0$ then α_1 is negative, while when $A < 0$ α_1 is positive (the same result can be obtained using Viète formulas for cubic equations) [110]. Additionally, the study of the first-order derivative shows that $\mathscr{F}(\alpha)$ has a maximum at $\alpha^* = 2B/(-3A)$ and a minimum at $\alpha = 0$. The trend of $\mathcal{F}(\alpha)$ is sketched in Fig. 2.

From our preliminary examination, it is possible to conclude that Eq. (38) is respected for any (positive) α if $A > 0$. This inequality translates into the following upper bound for β (cf. Eq. (40a)):

$$0 < \frac{\beta \xi_i}{r_0^2} < \frac{3}{\pi^3}\left(90\lambda \Sigma^2 r_0^2 - \pi^3\right), \tag{42}$$

which makes sense only if the quantity on the right is positive, i.e.,

$$\Sigma^2 r_0^2 > \frac{\pi^3}{90\lambda}, \tag{43}$$

where we recall that we are supposing $\lambda > 0$ (cf. Equation (36)). In other words, the WH turns out to be traversable when Eqs. (42) and (43) are satisfied, as these guarantee that Eq. (32), or equivalently Eq. (38), is fulfilled for any radial distance r.

Let us now consider the scenario where A is negative. In this case, the WH traversability is ensured if

$$0 \leq \alpha < \alpha_1, \tag{44}$$

and Eq. (42) no longer holds, as it is replaced by

$$\frac{\beta \xi_i}{r_0^2} > \frac{3}{\pi^3}\left(90\lambda \Sigma^2 r_0^2 - \pi^3\right), \tag{45}$$

while Eq. (43) is still valid.

At this point, we need to compute the real-valued root α_1 of the cubic (41) [110]. By introducing the change of variable

$$\alpha = y - \frac{B}{3A}, \tag{46}$$

we obtain the depressed cubic

$$y^3 + \mathcal{P}y + \mathcal{Q} = 0, \tag{47}$$

having coefficients

$$\mathcal{P} = -\frac{B^2}{3A^2}, \tag{48a}$$
$$\mathcal{Q} = \frac{2B^3 + 27A^2C}{27A^3}, \tag{48b}$$

and discriminant

$$\Delta = -\left(4\mathcal{P}^3 + 27\mathcal{Q}^2\right) = -\frac{C\left(27A^2C + 4B^3\right)}{A^4}, \tag{49}$$

which always attains negative values in view of Eq. (40). According to the general theory of algebraic third-order equations, this means that the original cubic (41) has only one real root, as pointed out before. Therefore, α_1 can be worked out via the Cardano formula jointly with Eq. (46), which gives

$$\alpha_1 = (u_1)^{1/3} + (u_2)^{1/3} - \frac{B}{3A}, \tag{50}$$

where

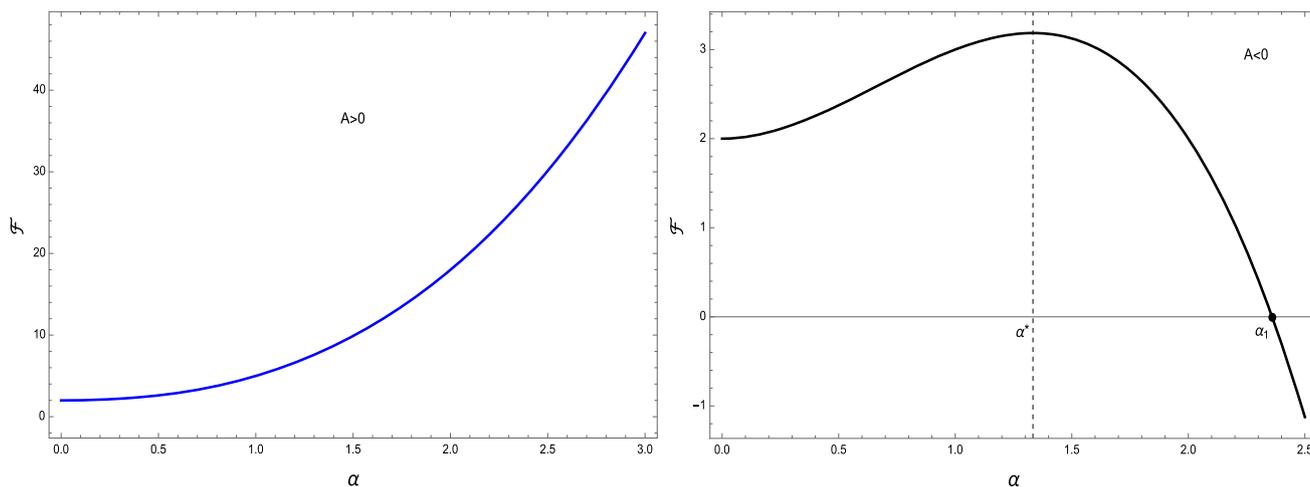$$u_{1,2} = -\frac{\mathcal{Q}}{2} \mp \frac{1}{6}\sqrt{-\frac{\Delta}{3}}. \tag{51}$$

It is now evident that α_1 is positive when $A < 0$, as both terms $-\mathcal{Q}/2$ and $-B/(3A)$ are positive in this case.

By performing a detailed investigation, we find that α_1 is always larger than one. In particular, provided that Eq. (45) is enforced, we have

$$\alpha_1 > \frac{3}{2}. \tag{52}$$

In addition,

$$\alpha_1 > n > 2^{2/3} \approx 1.59, \tag{53}$$

Fig. 2 Pictorial representation of the $\mathcal{F}(\alpha)$ function defined in Eq. (39) for $A > 0$ (left panel) and $A < 0$ (right panel). In the first case, the flaring-outward condition (32) is always satisfied, while in the second it holds when $\alpha < \alpha_1$. In this latter scenario, the point α^*, where $\mathcal{F}(\alpha)$ attains its maximum, is positive

if the GUP parameter β is subject to

$$0 < \frac{3}{\pi^3}\left(90\lambda\Sigma^2 r_0^2 - \pi^3\right) < \frac{\beta\xi_i}{r_0^2} < \frac{3n^2}{\pi^3\left(n^3 - 4\right)}$$
$$\times \left[90n\lambda r_0^2\Sigma^2 + \pi^3(2 - n)\right], \tag{54}$$

where we have taken into account the lower bound (45) (notice that the above relation is well-defined due to the fact that Eq. (43) is valid and $n > 2^{2/3}$). Therefore, thanks to Eqs. (52) and (53) we can conclude that the shape function always satisfies the flaring-outward condition (32) near the throat, which is thus not spoiled by the presence of β. This is a crucial result, which demonstrates that Casimir WHs incorporating GUP corrections can be formed in the RR gravity scenario.

4.2.2 First stability approach

The WH stability can be probed using a first stability approach that relies on the introduction of the (squared) adiabatic sound velocity [36,111,112]

$$v_s^2 = \frac{d\langle p\rangle}{dr}\left(\frac{d\rho}{dr}\right)^{-1}, \tag{55}$$

where $\langle p\rangle = \frac{1}{3}(p_r + 2p_t)$ is the pressure averaged across the three spatial dimensions. This quantity embodies the speed at which small disturbances or sound waves propagate through a medium in an adiabatic process (i.e., without the exchange of heat) and hence permits to investigate how small perturbations affect the WH geometry. Plugging Eqs. (20), (26), and

(27) into the formula (55), we find

$$v_s^2 = \frac{2}{3\lambda} - 1. \tag{56}$$

The WH stability follows from the enforcement of the causality condition

$$0 \leq v_s^2 < 1, \tag{57}$$

which ensures that v_s^2 is smaller than the speed of light, as required by the principles of special relativity. Owing to Eq. (56), this requirement is satisfied when

$$\frac{1}{3} < \lambda \leq \frac{2}{3}, \tag{58}$$

which agrees with the flaring-outward constraint (36), thus implying that the WH is both stable and traversable.

Interestingly, in Eq. (56), we have obtained the same result as in Ref. [69], which deals with generic WHs in RR gravity. Thus, by adopting this first paradigm, β does not influence the WH stability. The situation will change if a different stability criterion is used, as we will describe in the next section. This will allow us to show the pivotal role fulfilled by β in the evaluation of the WH properties. Hereafter, we will suppose Eq. (58).

4.2.3 Second stability approach

The second stability scheme is based on a mathematical method, first introduced in Ref. [113] and recently refined in Ref. [114], known as *Herrera cracking technique*. This procedure is used to identify the possible appearance of "crackings", i.e., breaks, in the fluid distributions of self-gravitating

compact objects caused by the presence of local arbitrarily small pressure anisotropies and density fluctuations. The formalism is suitable for analyzing also WH behaviour (see e.g. Refs. [115,116]) and its key result is that a stable system must satisfy specific constraints formulated in terms of the speed of sound. Indeed, let $\Gamma := p_t - p_r$ represent the local pressure anisotropy, then the relative order of magnitude of pressure and density perturbations give

$$\frac{\delta \Gamma}{\delta \rho} \sim \frac{\delta (p_t - p_r)}{\delta \rho} \sim \frac{\delta p_t}{\delta \rho} - \frac{\delta p_r}{\delta \rho} \sim V_t^2 - V_r^2, \tag{59}$$

where

$$V_t^2 = \frac{dp_t}{d\rho}, \tag{60}$$

$$V_r^2 = \frac{dp_r}{d\rho}, \tag{61}$$

denote the transverse and radial components of the (squared) sound velocity, respectively. The former quantity measures the response of the object to an outside pressure, while the latter the changes of pressure produced by an inside push. In our geometry, Eqs. (60) and (61) yield

$$V_t^2 = \frac{1}{8\lambda \left(1 + \frac{3\beta\xi_i}{2r^2}\right)} \left[\frac{3r}{r_0} - 8\lambda + 4r\beta\xi_i \left(\frac{1}{4r_0^3} + \frac{1 - 3\lambda}{r^3} \right) \right.$$
$$\left. - \frac{270\lambda\Sigma^2}{\pi^3} r r_0 \right], \tag{62}$$

$$V_r^2 = -\frac{1}{4\lambda \left(1 + \frac{3\beta\xi_i}{2r^2}\right)} \left[\frac{3r}{r_0} + 4(\lambda - 2) \right.$$
$$\left. + 2r\beta\xi_i \left(\frac{1}{2r_0^3} + \frac{3\lambda - 4}{r^3} \right) - \frac{270\lambda\Sigma^2}{\pi^3} r r_0 \right], \tag{63}$$

which, bearing in mind Eqs. (37) and (40), can be written equivalently as

$$V_t^2 = -\frac{1}{\lambda \left(\frac{4}{3}B\alpha^2 + 3C\right)} \left[A\alpha^3 + \frac{4}{3}\lambda B\alpha^2 + C(3\lambda - 1) \right], \tag{64}$$

$$V_r^2 = \frac{1}{\lambda \left(\frac{2}{3}B\alpha^2 + \frac{3}{2}C\right)} \left[A\alpha^3 + \frac{2}{3}(2 - \lambda) B\alpha^2 \right.$$
$$\left. + C \left(2 - \frac{3}{2}\lambda \right) \right]. \tag{65}$$

To avoid faster-than-light signals and the related causality violations, both V_t^2 and V_r^2 must fall within the interval $[0, 1)$, i.e.,

$$0 \le V_t^2 < 1, \tag{66a}$$

$$0 \le V_r^2 < 1, \tag{66b}$$

which implies $|V_t^2 - V_r^2| < 1$ [113]. Then, as demonstrated in Ref. [114], potentially unstable (resp. stable) regions occur within anisotropic matter configurations when the (squared) tangential speed of sound is larger (resp. smaller) than the radial one. Therefore, we can say that

$$-1 < V_t^2 - V_r^2 < 1 \Rightarrow$$
$$\begin{cases} -1 < V_t^2 - V_r^2 \le 0, & \text{WH is} \\ & \text{potentially stable,} \\ \\ 0 \le V_t^2 - V_r^2 < 1, & \text{WH is} \\ & \text{potentially unstable.} \end{cases} \tag{67}$$

Bearing in mind Eqs. (66) and (67), we find that the WH is stable if the following inequalities hold:

$$0 \le V_r^2 < 1, \tag{68a}$$

$$V_t^2 \ge 0, \tag{68b}$$

$$-1 < V_t^2 - V_r^2 \le 0. \tag{68c}$$

Starting from Eqs. (64) and (65), we arrive at a first fundamental conclusion: when $A > 0$, the WH is never stable, because both V_r^2 and V_t^2 do not satisfy the causality conditions (66). Therefore, we hereafter suppose $A < 0$ and the ensuing relations (43) and (45).

To investigate the WH stability, it is essential to figure out the behaviour of V_t^2 and V_r^2. A first idea can be obtained by considering Fig. 3, which provides a pictorial representation of these functions. Let us begin with the study of the derivatives, which gives

$$\frac{dV_t^2}{d\alpha} = -\frac{12\alpha A B}{\lambda \left(4\alpha^2 B + 9C\right)^2} \mathcal{G}(\alpha), \tag{69}$$

$$\frac{dV_r^2}{d\alpha} = \frac{24\alpha A B}{\lambda \left(4\alpha^2 B + 9C\right)^2} \mathcal{G}(\alpha), \tag{70}$$

where we have introduced the real-valued function

$$\mathcal{G}(\alpha) := \alpha^3 + \frac{27}{4} \frac{C}{B} \alpha + 2\frac{C}{A}. \tag{71}$$

Therefore, the monotonicity of V_t^2 and V_r^2 is ruled by the sign of $\mathcal{G}(\alpha)$. First of all, the depressed cubic $\mathcal{G}(\alpha) = 0$ admits one real-valued root, which we denote with $\tilde{\alpha}$, and the discriminant

$$\tilde{\Delta} = -\frac{27}{16} C^2 \left(\frac{64}{A^2} + \frac{729C}{B^3} \right). \tag{72}$$

Since $\tilde{\Delta}$ is negative, we can invoke the Cardano formula for evaluating $\tilde{\alpha}$, which thus assumes the form

$$\tilde{\alpha} = \left(-\frac{C}{A} + \frac{1}{6}\sqrt{-\frac{\tilde{\Delta}}{3}}\right)^{1/3} + \left(-\frac{C}{A} - \frac{1}{6}\sqrt{-\frac{\tilde{\Delta}}{3}}\right)^{1/3}. \tag{73}$$

In our hypotheses, $\tilde{\alpha}$ is positive. Then, from the analysis of the function $\mathcal{G}(\alpha)$, we realize that $\mathcal{G}(\alpha) < 0$ if $0 \leq \alpha < \tilde{\alpha}$, and $\mathcal{G}(\alpha) > 0$ when $\alpha > \tilde{\alpha}$. This fact permits to conclude that $\alpha = \tilde{\alpha}$ is a minimum for V_t^2 and a maximum for V_r^2. Moreover, as a consequence of Eq. (58) we have

$$v_t := V_t^2(\alpha = 0) = \frac{1}{3\lambda}(1 - 3\lambda) < 0, \tag{74}$$

$$v_r := V_r^2(\alpha = 0) = \frac{2}{3\lambda}\left(2 - \frac{3}{2}\lambda\right) \geq 1. \tag{75}$$

At this stage, let us calculate the explicit expressions for the points α_t and α_r where the tangential and radial sound velocities, respectively, vanish. These quantities can be found upon solving the cubics in square brackets in Eqs. (64) and (65). It follows from the the Descartes rule of signs that both α_t and α_r are positive, and, after a detailed analysis, we obtain

$$\alpha_t = -\frac{4\lambda}{9}\frac{B}{A} + \left(-\frac{\mathcal{Q}_t}{2} - \frac{1}{6}\sqrt{-\frac{\Delta_t}{3}}\right)^{1/3} + \left(-\frac{\mathcal{Q}_t}{2} + \frac{1}{6}\sqrt{-\frac{\Delta_t}{3}}\right)^{1/3}, \tag{76}$$

$$\alpha_r = \frac{2}{9}\frac{B}{A}(\lambda - 2) + \left(-\frac{\mathcal{Q}_r}{2} - \frac{1}{6}\sqrt{-\frac{\Delta_r}{3}}\right)^{1/3} + \left(-\frac{\mathcal{Q}_r}{2} + \frac{1}{6}\sqrt{-\frac{\Delta_r}{3}}\right)^{1/3}, \tag{77}$$

with

$$\begin{aligned}
\mathcal{Q}_t &= \frac{C(3\lambda - 1)}{A} + \frac{128B^3\lambda^3}{729A^3}, \\
\Delta_t &= \frac{C(1 - 3\lambda)}{27A^4}\left[729A^2C(3\lambda - 1) + 256B^3\lambda^3\right], \\
\mathcal{Q}_r &= \frac{C(4 - 3\lambda)}{2A} - \frac{16B^3(\lambda - 2)^3}{729A^3} < \mathcal{Q}_t, \\
\Delta_r &= \frac{C(4 - 3\lambda)}{108A^4}\left[729A^2C(3\lambda - 4) + 64B^3(\lambda - 2)^3\right] \\
&\quad < \Delta_t.
\end{aligned} \tag{78}$$

Let us now determine the intersection point α_i of the functions V_t^2 and V_r^2. Starting from Eqs. (64) and (65), we get

$$V_t^2 - V_r^2 = -\frac{9}{\lambda\left(4B\alpha^2 + 9C\right)}\mathcal{G}_v(\alpha), \tag{79}$$

where

$$\mathcal{G}_v(\alpha) := A\alpha^3 + \frac{8}{9}B\alpha^2 + C. \tag{80}$$

Therefore, α_i represents the positive real-valued solution of the cubic $\mathcal{G}_v(\alpha) = 0$, which yields

$$\alpha_i = -\frac{8B}{27A} + \left(-\frac{\mathcal{Q}_i}{2} - \frac{1}{6}\sqrt{-\frac{\Delta_i}{3}}\right)^{1/3} + \left(-\frac{\mathcal{Q}_i}{2} + \frac{1}{6}\sqrt{-\frac{\Delta_i}{3}}\right)^{1/3}, \tag{81}$$

with

$$\begin{aligned}
\mathcal{Q}_i &= \frac{1}{27A^3}\left(27A^2C + \frac{1024B^3}{729}\right), \\
\Delta_i &= -\left(\frac{2048B^3C}{729A^4} + \frac{27C^2}{A^2}\right).
\end{aligned} \tag{82}$$

We are now ready to address the question of the WH stability. First of all, the physically meaningful solutions of the system (68) are those for which $\alpha_r > \alpha_t$, i.e., α_r must be strictly larger than α_t. For if this relation were not true, then V_r^2 and V_t^2 would not satisfy causality criteria (66) at the same time. In our hypotheses, the inequality $\alpha_r > \alpha_t$ is satisfied if

$$-\frac{\mathcal{Q}_r}{2} - \frac{1}{6}\sqrt{-\frac{\Delta_r}{3}} > -\frac{\mathcal{Q}_t}{2} - \frac{1}{6}\sqrt{-\frac{\Delta_t}{3}}, \tag{83}$$

which gives

$$\Delta_r - \Delta_t + 3(\mathcal{Q}_r - \mathcal{Q}_t)\left(9\mathcal{Q}_r - 9\mathcal{Q}_t - 2\sqrt{3}\sqrt{-\Delta_t}\right) > 0. \tag{84}$$

A necessary but not sufficient condition for the above relation to hold is that $9\mathcal{Q}_r - 9\mathcal{Q}_t - 2\sqrt{3}\sqrt{-\Delta_t} < 0$, a constraint that is always fulfilled in our hypotheses.[1] Given these premises, a key result is that (cf. Eq. (75))

$$V_r^2(\alpha = \tilde{\alpha}) \geq v_r \geq 1, \tag{85}$$

---

[1] We observe that Eq. (83) is automatically true if Eqs. (42) and (43), i.e. the conditions for $A > 0$, are valid. However, we know that when $A > 0$ the WH cannot be stable.

Fig. 3 A pictorial sketch of the functions V_t^2 and V_r^2, defined in Eqs. (64) and (65)
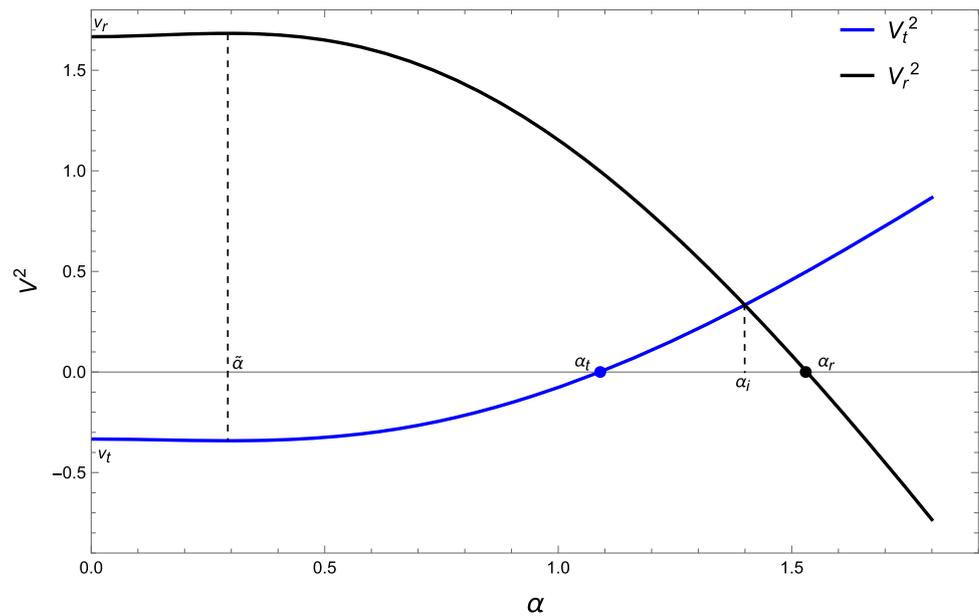
which translates into the fact that V_r^2 complies with the causality condition (68a) only in the interval

$$\tilde{\alpha} < \alpha \le \alpha_r. \tag{86}$$

Moreover, the behaviour of the function $\mathcal{G}_v(\alpha)$ occurring in Eq. (79) permits to figure out another crucial point. Indeed, $\mathcal{G}_v(\alpha)$ is positive when $0 \le \alpha < \alpha_i$ and attains a maximum at $\alpha = -16B/(27A)$. Therefore, we can claim that $V_t^2 - V_r^2 \le 0$ when

$$0 \le \alpha \le \alpha_i. \tag{87}$$

Therefore, bearing in mind Eqs. (86) and (87), we can conclude that the system (68) is satisfied if (see also Fig. 3)

$$\tilde{\alpha} < \alpha \le \alpha_r, \tag{88a}$$
$$\alpha \ge \alpha_t, \tag{88b}$$
$$0 \le \alpha \le \alpha_i. \tag{88c}$$

Now, since $\alpha_t < \alpha_i < \alpha_r$ and assuming that $\alpha_r > \alpha_t$ (which implies that Eq. (83), or equivalently Eq. (84), is valid), the system (88) simplifies to

$$\alpha_t \le \alpha \le \alpha_i. \tag{89}$$

Therefore, we can claim that the WH is stable (and traversable, see below) if Eq. (89) is met. Moreover, α_i is subject to the key relation

$$\alpha_i > \frac{3}{2}, \tag{90}$$

which proves that in the vicinity of the throat the WH is stable. Notice that also α_1 satisfies an analogous inequality (see Eq. (52)).[2]

In order to investigate the WH traversability and stability, Eq. (89) should be compared with the constraint (44). Bearing in mind formulas (50) and (81), we arrive at the fundamental inequality

$$\alpha_1 > \alpha_i, \tag{91}$$

which leads to the crucial conclusion that the GUP-corrected Casimir WH can be both traversable and stable. Indeed, the validity of Eq. (91) implies that when the WH is stable (i.e., Eq. (89) is valid) then it is also traversable (i.e., Eq. (44) holds), but the converse is not true. This result is also physically reasonable, as the requirement of stability reduces the allowed range for α from the original interval (44), where the flaring-outward condition (32) is respected, to the stricter one (89). This reflects the fact that stability is an additional requirement that further constrains the WH geometry.

It is worth noticing that $\alpha_1 < \alpha_r$ when $1/3 < \lambda < 1/2$, while $\alpha_1 > \alpha_r$ for $1/2 \le \lambda \le 2/3$. Then, owing to Eq. (91), the situation is the following:

$$0 < \tilde{\alpha} < \alpha_t < \alpha_i < \alpha_1 < \alpha_r, \quad \text{if } \frac{1}{3} < \lambda < \frac{1}{2}, \tag{92}$$

---

[2] It is interesting to note that in general, given any integer $m > 1$, the condition $\alpha_i > m$ is satisfied according to the pattern

$$\begin{cases} 9Am^3 + 8Bm^2 + 9C > 0, & \text{if } m \bmod 3 \ne 0, \\ Am^3 + 8B\left(\frac{m}{3}\right)^2 + C > 0, & \text{if } m \bmod 3 = 0. \end{cases}$$

and

$$0 < \tilde{\alpha} < \alpha_t < \alpha_i < \alpha_r < \alpha_1, \quad \text{if } \frac{1}{2} \le \lambda \le \frac{2}{3}, \quad (93)$$

where we have assumed Eq. (83), or equivalently Eq. (84), to suppose that $\alpha_t < \alpha_r$.

To further stress how the GUP parameter β influences the WH stability and traversability, in the next section we will investigate Casimir WHs in the regime $\beta = 0$. Interesting differences with the model analyzed so far will emerge.

4.2.4 The case $\beta = 0$

The model with $\beta = 0$ refers to Casimir WHs involving no GUP corrections. In this context, starting from Eq. (25), we obtain

$$(b - rb')_{\beta=0} = r_0 - \frac{\pi^3(\alpha - 2)}{90\lambda\alpha r_0 \Sigma^2}, \quad (94)$$

and hence the flaring-outward condition (32) is secured, in the hypothesis $\lambda > 0$, if either the constraint (43) is valid or the following relations are met:

$$\begin{cases} 90\lambda r_0^2 \Sigma^2 < \pi^3, \\ \alpha < \frac{2\pi^3}{\pi^3 - 90\lambda r_0^2 \Sigma^2}. \end{cases} \quad (95)$$

The (squared) adiabatic sound velocity retains the same value as in Eq. (55), which means that the inequality (58) guarantees WH stability according to the first stability approach outlined in Sect. 4.2.2. Following the recipe of the second stability method described in Sect. 4.2.3 and upon assuming Eq. (58), we can now calculate the transverse and radial sound velocities. Starting from Eqs. (60) and (61), we arrive at the following formulas:

$$\left(V_t^2\right)_{\beta=0} = \frac{3}{8}\alpha\left(\frac{1}{\lambda} - \frac{90 r_0^2 \Sigma^2}{\pi^3}\right) - 1,$$

$$\left(V_r^2\right)_{\beta=0} = \frac{1}{\lambda}\left(2 - \frac{3\alpha}{4}\right) + \frac{135\alpha r_0^2 \Sigma^2}{2\pi^3} - 1, \quad (96)$$

which permit to achieve a first interesting conclusion: if Eq. (43) holds, then the WH is traversable but not stable, as the stability requirements (68) are not satisfied. This is in sharp contrast to the scenario with $\beta \ne 0$, where Eq. (43) does not compromise either WH stability or traversability. On the other hand, in the setup with $\beta = 0$, the WH is both stable and traversable if $\lambda < 2/3$ and

$$\frac{8\pi^3\lambda}{3\left(\pi^3 - 90\lambda r_0^2 \Sigma^2\right)} \le \alpha \le \frac{16\pi^3}{9\left(\pi^3 - 90\lambda r_0^2 \Sigma^2\right)}, \quad (97)$$

which is consistent with Eq. (95). This situation resembles the previous framework (see Eq. (89)), albeit with different assumptions (recall that Eq. (43) does not imply stability when $\beta = 0$). Our analysis thus reveals the remarkable point that the factor β can completely alter the criteria for WH stability. In other words, the fact that the presence of a nonvanishing GUP parameter does not spoil either the traversability or the stability of Casimir WHs is not trivial, as it is not something to be expected automatically from the beginning.

4.3 The emergence of exotic matter

As pointed out before, the occurrence of exotic matter in WH spacetimes implies the breach of the NEC. We now determine whether the GUP-corrected Casimir WHs in the RR gravity theory adhere to the NEC, which, we recall, translates into the relations $\rho(r) + p_r(r) \ge 0$ and $\rho(r) + p_t(r) \ge 0$ [2].

Bearing in mind Eq. (12a) (or equivalently Eq. (20)) and Eqs. (26) and (27), we obtain

$$\rho + p_r = -\frac{\mathcal{F}(\alpha)}{2160\pi\lambda r_0^4 \alpha^6}, \quad (98)$$

$$\rho + p_t = \frac{1}{4320\pi\lambda r_0^4 \alpha^6}\left(A\alpha^3 - \frac{C}{2}\right), \quad (99)$$

where the $\mathcal{F}(\alpha)$ function and its coefficients have been defined in Eqs. (39) and (40), respectively. It follows from Eq. (98) that as soon as the flaring-outward condition (32) is met, $\rho + p_r$ is negative owing to Eq. (38). Furthermore, we recall that, according to the analysis of Sect. 4.2.3, the WH is stable only in the case $A < 0$, and, in this hypothesis, $\rho + p_t$ is also negative. Since we have proved that WH stability implies traversability (see comments below Eq. (91)), we can conclude that traversable and stable GUP-corrected Casimir WHs always require the existence of NEC-violating fields.

The total amount $\mathcal{M}$ of exotic matter residing in the spacetime can be calculated via the volume integral [117] (see also e.g. Ref. [75])

$$\mathcal{M} = \oint (\rho(r) + p_r(r))\,dV = 2\int_{r_0}^{\infty} (\rho(r) + p_r(r))\,dV$$

$$= 8\pi\int_{r_0}^{\infty} (\rho(r) + p_r(r))\,r^2 dr, \quad (100)$$

where we have used the relation $\oint dV = 2\int_{r_0}^{\infty} dV = 8\pi\int_{r_0}^{\infty} r^2 dr$. By introducing a cutoff parameter $\mathcal{R}$, we thus obtain

$$\mathcal{M} = 8\pi \int_{r_0}^{\mathcal{R}} [\rho(r) + p_r(r)] \, r^2 dr$$

$$= 8\pi \int_{1}^{\alpha_{\mathcal{R}}} [\rho(\alpha) + p_r(\alpha)] \, r_0^3 \alpha^2 d\alpha$$

$$= -\frac{1}{810\alpha_{\mathcal{R}}^3 \lambda r_0} \Big[\alpha_{\mathcal{R}}^3 (3A \log \alpha_{\mathcal{R}} + 3B + C)$$

$$- 3B\alpha_{\mathcal{R}}^2 - C \Big], \tag{101}$$

with $\alpha_{\mathcal{R}} = \mathcal{R}/r_0$. Notably, the GUP factor significantly influences the exotic mass content. To demonstrate this point, we first evaluate the behaviour of $\mathcal{M}$ near the throat by expanding the formula (101) around $\alpha_{\mathcal{R}} = 1$:

$$\mathcal{M} = -\frac{\pi^3 + 90\lambda \Sigma^2 r_0^2 + \pi^3 \beta \xi_i / r_0^2}{90\lambda r_0} (\alpha_{\mathcal{R}} - 1)$$

$$+ O\left[(\alpha_{\mathcal{R}} - 1)^2 \right], \tag{102}$$

and then compare it with the corresponding quantity computed in the regime $\beta = 0$:

$$(\mathcal{M})_{\beta=0} = -\frac{\pi^3 + 90\lambda \Sigma^2 r_0^2}{90\lambda r_0} (\alpha_{\mathcal{R}} - 1) + O\left[(\alpha_{\mathcal{R}} - 1)^2 \right]. \tag{103}$$

We thus see that, in a region close to the throat, the following inequality holds:

$$|\mathcal{M}| > |(\mathcal{M})_{\beta=0}|, \tag{104}$$

which entails that the amount of exotic matter sourcing GUP-corrected Casimir WHs increases as the value of β grows. This is not a novel feature in this research field, as exotic matter exhibits a similar trend also in Ref. [77], which deals with GUP-modified Casimir WHs in symmetric teleparallel gravity. Such result is physically reasonable, since the larger is β the greater is the modulus of the Casimir energy density $\rho(r)$ (cf. Eq. (20)), implying a more significant presence of NEC-violating fields. Thus, our findings confirm that the GUP-modified Casimir energy can serve as a source of exotic matter, which remains a key element for the formation and maintenance of stable WHs.

4.4 Light deflection angle

In this section, we investigate the phenomenon of light deflection in the WH gravitational field.

It follows from Eq. (10) that the orbit of a photon in the equatorial plane of the WH spacetime is characterized by the relation

$$dt^2 = \frac{1}{1 - \frac{b(r)}{r}} dr^2 + r^2 d\phi^2, \tag{105}$$

where we have introduced the coordinate rescaling $e^{2\Phi(r)} dt^2 \to dt^2$ taking advantage of the fact that for a zero-tidal-force WH $e^{2\Phi(r)} = const.$

In order to compute the deflection angle Θ of a light ray passing by the WH, we employ a recent geometric approach that relies on the Gauss-Bonnet theorem [118]. Thanks to this method, Θ can be written via the integral expression

$$\Theta = -\int_0^\pi \int_{r=\frac{\ell}{\sin\phi}}^\infty \mathcal{K} dS, \tag{106}$$

where $\mathcal{K}$ is the Gaussian optical curvature, dS the optical surface element, and ℓ the impact parameter, which, to first approximation, can be evaluated as the distance of closest approach to the WH center. By rearranging the optical metric (105) as

$$dt^2 = du^2 + \mathcal{H}^2(u) d\phi^2, \tag{107}$$

where $\mathcal{H} = r$ and the new coordinate u satisfies

$$du = \frac{1}{\sqrt{1 - \frac{b(r)}{r}}} dr, \tag{108}$$

we can write $\mathcal{K}$ through the general formula [119] (see also Eq. (47) in Ref. [120])

$$\mathcal{K} = -\frac{1}{\mathcal{H}(u)} \left[\frac{dr}{du} \frac{d}{dr} \left(\frac{dr}{du} \right) \frac{d\mathcal{H}}{dr} + \left(\frac{dr}{du} \right)^2 \frac{d^2\mathcal{H}}{dr^2} \right], \tag{109}$$

which in our geometry yields

$$\mathcal{K} = -\frac{r_0}{2r^3} + \frac{\pi^3}{540\lambda \Sigma^2 r_0^3 r^6} \Big[3r^2 r_0^2 (r - 2r_0)$$

$$+ \beta \xi_i \left(r^3 - 4r_0^3 \right) \Big]. \tag{110}$$

We can thus work out the deflection angle by plugging Eq. (110) into the expression (106) and then computing the ensuing integral. In this way, we find the exact result

$$\Theta = \frac{r_0}{\ell} - \frac{\pi^3}{90\ell r_0 \lambda \Sigma^2} \left[1 - \frac{\pi r_0}{4\ell} + \beta \xi_i \left(\frac{1}{3r_0^2} - \frac{\pi r_0}{16\ell^3} \right) \right]. \tag{111}$$

Apart from the contributions associated with the WH geometry, the deflection angle is influenced by the GUP

Fig. 4 Deflection angle Θ as a function of the impact parameter ℓ for the KMM model with $r_0 = 1.7$, $\lambda = 0.34$, $\Sigma = 1.7$, and β attaining five different values: $\beta = 0, 0.4, 1.45, 2.5, 10$. The behaviour of Θ varies depending on whether β is zero or non-zero
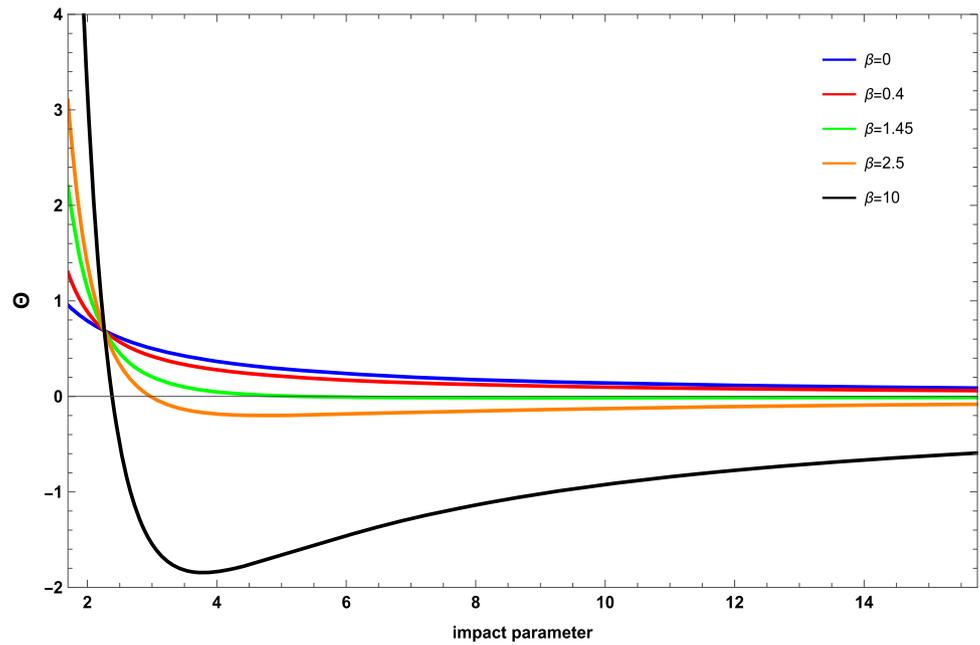
parameter β, as indicated by the last term in Eq. (111). Moreover, our formula for Θ reduces to the corresponding GR relation pertaining to zero-tidal-force GUP-corrected Casimir WHs in the limit $\lambda = \Sigma^2 = 1$ (see Ref. [75] for further details).

The deflection angle goes to zero as the impact parameter approaches infinity, indicating that the trajectories of photons moving far away from the WH remain almost unaffected. Conversely, the expansion of Eq. (111) around $\ell = r_0$ yields

$$
\begin{aligned}
\Theta = 1 - \frac{\pi^3}{90\lambda r_0^2 \Sigma^2} & \left[1 - \frac{\pi}{4} + \frac{(16 - 3\pi)\beta\xi_i}{48r_0^2} \right] \\
& - \left\{ \frac{1}{r_0} + \frac{\pi^3}{1080r_0^5\lambda\Sigma^2} \left[6r_0^2(\pi - 2) + \beta\xi_i(3\pi - 4) \right] \right\} \\
& \times (\ell - r_0) + O\left[(\ell - r_0)^2 \right],
\end{aligned}
\tag{112}
$$

and hence Θ can be either positive or negative when ℓ takes values close to the WH throat. In the first scenario, light rays fall toward the WH, while, in the second, they bend outward (recall however that to describe the light bending one should consider the absolute value of Θ). Interestingly, there exist situations where the features of Θ can be somewhat modified by the presence of a nonvanishing GUP parameter β. This is evident from Fig. 4, which shows that in the regime $\beta = 0$ the deflection angle is always positive, while it can assume negative values when β is nonvanishing.

5 Concluding remarks

In this paper, we have studied static and spherically symmetric GUP-modified Casimir WHs within the context of RR gravity, which arises from incorporating the energy-dependent metric featuring Rainbow scenario into the Rastall model (see Sect. 2). These objects are supported by the negative Casimir energy density containing GUP contributions (see Sect. 3) and naturally allow for a quantum-gravity-inspired NEC violation, a possibility that makes them particularly relevant to the current theoretical literature. The GUP has been formulated to modify the Heisenberg uncertainty principle via the deforming factor β, which is expected to emerge from candidate theories of quantum gravity. GUP corrections alter the overall energy–momentum distribution of matter sourcing WHs, thereby affecting their geometric properties. This means that the analysis of WHs comprising GUP adjustments offers precious insights into the role that quantum gravity can assume in the formation of nontrivial spacetime structures.

In the realm of WHs, three essential questions stand out: traversability, stability, and the existence of exotic matter. We have provided an original and thorough examination of these aspects in Sect. 4. In particular, we have tackled the stability issue by employing two different paradigms, with the adiabatic sound velocity as the primary physical quantity considered. Notably, we have utilized the so-called Herrera cracking technique, which, despite its elegance and effectiveness, appears to be not widely harnessed in the WH research field. To the best of our knowledge, while the original scheme developed in Ref. [113] has been somewhat explored in WH studies (see e.g. Refs. [115,116]), its refinement recently worked out in Ref. [114] has not yet been applied to WH stability analysis. Unlike the approach discussed in Sect. 4.2.2, where the GUP term β plays no role in the assessment of the WH stability, the (improved) Herrera cracking procedure has the advantage of highlighting the crucial contribution of

β. In fact, by means of this formalism, we have established that there are always values of β that guarantee stability, a requirement that in our setup also ensures the fulfillment of the flaring-outward constraint (see Sect. 4.2.3). In other words, our investigation has allowed us to demonstrate that Casimir WHs framed within the RR pattern can achieve both stability and traversability. Moreover, we have found that these properties are inherently linked to the occurrence of NEC-violating matter, whose amount increases as β grows.

The proof that Casimir WHs can be stable is not a result to be taken for granted, as it could potentially be spoiled by the presence of the GUP parameter. In fact, as outlined in Sect. 4.2.4, the stability criteria differ based on whether β is zero or nonzero. This fact shows that WHs incorporating GUP corrections are fascinating solutions which deserve further consideration. The main claim of this paper is that such WHs can indeed arise within RR theory. This represents a key finding especially if we compare it with the outcome of Ref. [69], where it is proved that generic zero-tidal-force WHs cannot be formed in RR framework upon considering specific parameter combinations and equations of state.

Finally, another fundamental facet of this paper is the application of the revised Herrera approach and the ensuing stability conditions that we have computed in Eq. (68). Given their broad applicability, we deem that the plethora of WH prototypes devised in the literature should be analyzed exploiting this method to assess whether they are stable. In this way, it is likely that valuable perspectives in this field will be revealed.

Data Availability Statement This manuscript has no associated data. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.].

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