mathematics

Article

Methods of Multi-Criteria Optimization of Technological Processes in a Fuzzy Environment Based on the Simplex Method and the Theory of Fuzzy Sets

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Abstract: Many modern technological objects in practice are characterized by the uncertainty of the initial information necessary for their management. Recently, one of the pressing scientific and practical problems is the development of new optimization methods for controlling the operating modes of such objects in a fuzzy environment. In this regard, the objective of this study is to develop methods of multi-criteria optimization in a fuzzy environment by modifying the simplex method and various optimality principles based on fuzzy mathematics methods. The methodology of the proposed study is based on a hybrid approach, which consists of the integrated use and modification of simplex methods and optimization methods with various optimality principles for working in a fuzzy environment. The main results are as follows: a simplex method of multi-criteria optimization of immeasurable criteria (here, we are talking about the impossibility of physical measurements of criteria, the values of which are estimated by decision maker); a theorem on the convergence of the solution sequence obtained using the proposed method to the minimum value of the criteria; a heuristic method based on a modification for fuzziness and a combination of the maximin and Pareto optimality principles, which allows effectively solving multi-criteria optimization problems in a fuzzy environment. The heuristic method proposed will be used to solve a real production problem—optimization of the technological process of benzene production.

Keywords: decision maker (DM); fuzzy constraints; heuristic method; multi-criteria optimization; simplex method

MSC: 49N30; 76B75; 90B50

Citation: Orazbayev, B.; Orazbayeva, K.; Ospanov, Y.; Suleimenova, S.; Kurmangaziyeva, L.; Makhatova, V.; Izbassarov, Y.; Otebaeva, A. Methods of Multi-Criteria Optimization of Technological Processes in a Fuzzy Environment Based on the Simplex Method and the Theory of Fuzzy Sets. *Mathematics* **2024**, *12*, 2856. https://doi.org/10.3390/math12182856

Academic Editor: Michael Voskoglou

Received: 15 August 2024
Revised: 10 September 2024
Accepted: 11 September 2024
Published: 14 September 2024

1. Introduction

When managing complex technological objects and processes of various industries, the problems of ensuring maximum efficiency and quality of decisions made can be formulated in the form of a multi-criteria optimization problem, which is solved by optimization methods.

In solving such problems, search optimization methods that use local information about the properties of the optimization object have recently played a special role. Such methods are iterative and make it possible to consistently improve the quality of solutions to optimization problems under conditions of uncertainty and fuzziness of the initial information. In practice, it is often impossible or very difficult to determine the dependence of the optimized function on the process parameters. In such conditions, the simplex search optimization method based on the use of information about the values of the optimized function is of great practical importance for solving real production problems [1–3]. When solving such optimization problems with immeasurable criteria, it will be necessary to involve a decision maker (DM)—an operator–technologist who makes decisions on the management of technological objects—and experts in the subject area who fuzzily evaluate the values of the criteria based on their knowledge, experience, and intuition.

At present, the solution of problems of multi-criteria optimization of technological processes with fuzzily described criteria is of great importance for modern science and practice. The relevance of solving such optimization problems is justified by the complication and increase in the number of technological objects and processes, during optimization of which it is necessary to take into account many contradictory criteria. Moreover, these contradictory criteria can be in the form of mathematical expressions that can be calculated but may not be formalized; therefore, they are assessed by a DM and experts fuzzily in natural language. Traditional methods of optimization theory are mainly focused on solving single-criterion optimization problems in deterministic conditions. Thus, when solving real optimization problems of various industries, it is necessary to take into account many criteria and restrictions that are characterized by fuzziness. In this regard, the use of traditional optimization methods for solving such optimization problems of real industries in a fuzzy environment is ineffective or unsuitable. Accordingly, there is a need to improve and adapt them or develop new optimization methods that can effectively solve such optimization problems in conditions of fuzzy criteria. When solving a production problem of multi-criteria optimization, the criteria are usually economic and environmental indicators of production, volume, and quality of products. These criteria are also often characterized by inconsistency, i.e., improving one of them can lead to the deterioration of the other. Therefore, solving a multi-criteria optimization problem is reduced to solving a decision-making problem in a fuzzy environment, the essence of which is to choose the most effective compromise solution in the current situation from the area of effective solutions (Pareto set) [4–7].

2. Literature Review

Let us present the results of an analysis of research works devoted to the study and solution of multi-criteria optimization problems in various conditions. The founder of fuzzy set theories, Zadeh, in his research [8,9], proposed a methodology for making approximate decisions and reasoning about a linguistic variable. Aliev et al., in their monograph [10], investigated and provided examples of production management in a fuzzy environment. Deb et al., in the works [11,12], explored the fundamental principles of multi-criteria optimization and described the main classical methods based on reducing a multi-chip problem to a single-criterion one and evolutionary methods of multi-criteria optimization. The importance of multi-criteria optimization in practice is substantiated, and key resources in the field of multi-criteria optimization are proposed. He et al., in studies [13–16], proposed an improved method for determining the best solution and an evolutionary method based on the Pareto principle of optimality.

The authors of [17,18] reviewed the concepts, methods of multi-criteria optimization, and methods for multi-criteria decision-making for solving specific practical problems. The book edited by Stadler [19] explores the issues and features of solving multi-criteria optimization problems in Engineering and in the Sciences. The author of article [20] studied multi-criteria problems for a special class of prefractal graphs and proposed an approach to solving them. The problem of multi-criteria optimization of a dynamic system and an approach to solving them based on methods of similarity theory and the importance of criteria is presented in the work [21]. Chłopińska et al., in studies [22,23], describe a method for solving multi-criteria optimization of the distribution of liquefied natural gas for engines based on genetic algorithms, and the authors of [24] used multi-criteria optimization methods to solve problems of distribution of humanitarian aid. Issues of using multi-criteria optimization methods in the process of calibrating digital measuring instruments were studied by the authors of the work [25].

Verdegay et al. investigated the application of fuzzy optimization in operational research and for solving other problems [26,27]. Ramik et al., in their works [28], proposed a fuzzy optimization method based on the concept of ideal and anti-ideal points to solve multi-criteria fuzzy optimization problems. Fu, in article [29], based on the principle of ideal and anti-ideal points, solved a multi-criteria optimization problem for reservoir flood management, demonstrating the effectiveness of the proposed method. The authors of the works [30–34] studied the theory and methods of fuzzy optimization, their complexity, and the computational method of fuzzy optimization, in which the promising direction of fuzzy mathematical programming and possible ways of further development are discussed.

An approach to solving fuzzy programming problems and a genetic algorithm based on fuzzy modeling is described by Iwamura in the book [35]. Possible directions of development and application of fuzzy mathematical programming methods for solving practical problems are discussed in the book [36]. The authors of works [37–40] studied one of the most well-known types of problems and methods of fuzzy mathematical programming—fuzzy linear programming (FLP). In works [37,38], FLP with interactive uncertain parameters and linear programming with fuzzy coefficients in constraints were studied. Conditional FLP problems and an approach to solving them based on the penalty function method were proposed by Jamison et al. in [39]. Rommelfanger, in [40], presented FLP models with soft constraints, described FLP problems in which the constraint coefficients and/or objective function can be fuzzy, and considered practical applications of FLP methods.

The authors of [41,42] proposed decision-making techniques for evaluating alternatives to renewable energy sources and interactive fuzzy optimization. Tian et al., in [43], proposed a MATLAB platform for evolutionary multi-criteria optimization. In [44,45], the authors proposed fuzzy multi-criteria decision-making methods and issues of its application in practice. In [44], linguistic values were used to evaluate the effectiveness of fuzzy criteria, and fuzzy numbers were used to evaluate the effectiveness of quantitative criteria for each task, taking into account its specific conditions. The authors of [45] used the multi-criteria decision-making method to optimize flood control operations. A multi-criteria decision-making model for making decisions for reservoir flooding process control is proposed in the article [46].

In recent years, one of the effective approaches to solving the problems of multi-criteria and fuzziness of optimization criteria is an approach based on the use of experience, knowledge, intuition, and preference of the DM to make the best decision. The authors of the study [47] presented an overview of research into methods for conducting the Multiple Criteria Decision Analysis process, which forms the basis of decision support systems (DSS). Fundamental works [48,49] outlined the foundations of theories and methods of decision-making and procedures for obtaining information from DMs in the decision-making process. At the same time, the information from DMs and experts used in the process of solving the problem is expressed fuzzily in natural language. Receipt, processing, formalization, and use of such fuzzy information are carried out on the basis of expert assessment methods and fuzzy set theories [50–53].

For a clearer presentation of the problem of multi-criteria optimization in a fuzzy environment, we note that recent sources, for example, [54–56], consider approaches to solving fuzzy optimization problems. These and other known approaches to fuzzy optimization are based on the transformation of the original fuzzy problem to a set of crisp problems. In this case, to transform a fuzzy problem into approximate crisp problems, the α-level set of fuzzy set theory is mainly used [57,58]. But unfortunately, in these methods, a considerable part of the collected fuzzy information is lost, which is the knowledge, experience, and intuition of the DM and experts, which is of great importance. Accordingly, the adequacy and effectiveness of the obtained solutions decrease, which is the main drawback of the known methods for solving fuzzy problems.

Based on the results of the literature analysis on solving multi-criteria optimization problems in a fuzzy environment, it can be noted that existing studies mainly consider a single-criterion case and do not take into account the DM preferences in the decision-making process. Multi-criteria problems are mainly solved by reducing many criteria to one integrated criterion by convolving them. Then, one of the well-known methods for solving single-criterion problems is applied. But with different physical contents of the criteria, when they are measured in different units of measurement, their integration into one criterion is impossible or impossible; accordingly, this approach is not applicable; application will lead to an incorrect decision. Another obstacle to solving problems of optimization of real objects and processes using known methods is their difficulty in formalizing, complexity, or the impossibility of building models that describe the dependence of criteria on input and operating parameters.

In works aimed at solving optimization problems in a fuzzy environment, as a rule, based on the α-level set, the original fuzzy problem is represented by a set of crisp problems on α slices. Then, the resulting set of crisp problems is solved using suitable deterministic optimization methods, and by combining the resulting solutions, using the formula of fuzzy set theories, the solution to the original fuzzy problem is determined. This approach allows solving a fuzzy optimization problem but leads to the loss of part of the collected fuzzy information, i.e., experience, knowledge, and intuition of DMs and experts, which in turn reduces the adequacy of the resulting solution. Although it is possible to increase the adequacy of the resulting solution by increasing the number of α slices, the dimension of the problem sharply increases, which reduces the effectiveness of this approach to solving fuzzy problems.

In this regard, currently, the development, improvement, and development of optimization methods for effectively solving multi-criteria optimization problems in a fuzzy environment is an urgent task of science and practice.

This state-of-the-art approach to solving multi-criteria optimization problems with fuzzy elements requires the development of new or modification of existing methods of multi-criteria and fuzzy optimization, which significantly motivates the work performed in this study.

This state-of-the-art method for solving multi-criteria fuzzy optimization problems requires the development of new or modification of existing methods of multi-criteria and fuzzy optimization, which significantly motivates the work performed in this study.

The essence of this study is to modify the idea of the simplex method to multi-criteria optimization with a vector of criteria fuzzily estimated by the DM and to develop a heuristic method of multi-criteria fuzzy optimization, providing more adequate, effective solutions. These results aimed at effectively solving the problem of multi-criteria optimization with fuzzy elements based on the modification of the simplex method and maximum use of fuzzy information and DM preferences based on the heuristic method are sufficient motivations for this work.

Further, in the following sections, the article is organized as follows.

Section 2, Materials and Methods, describes the object of study and characterizes the data used and materials and methods used in this study. In addition, this section proposes the simplex method adapted for solving a multi-criteria optimization problem with fuzzy criteria and a heuristic method of multi-criteria optimization in a fuzzy environment based on the modification of various principles of optimality.

The formulated theorem on the convergence of the proposed modified simplex method with proof, the formulation of a specific multi-criteria problem of optimizing the benzene production process with fuzzy constraints, and the results of its solution based on the proposed heuristic method are provided in Section 3, Results. This section also compares the optimization results based on the proposed heuristic method and the known deterministic method, shows the advantages of the proposed method, and provides graphs of the results.

The Discussion and Conclusions sections discuss the results obtained and provide a conclusion on the main results, consider the novelty of the study, as well as possible limitations of the proposed heuristic method and ways to eliminate and mitigate them.

3. Material and Methods

The object of this study is the technological processes of oil refining production, for example, the production of benzene and sulfur, taking place in technological installations of the Atyrau refinery, which is difficult to formalize and is characterized by the fuzziness of some parameters. The sulfur production plant is designed to produce sulfur from acidic wastewater and waste gases of oil refining plants through the adsorption of sulfur by an amine solution based on catalytic conversion into crystalline sulfur. To optimize the operating modes of such objects and the processes occurring in them, it is necessary to develop and apply effective multi-criteria optimization methods that are operational in a fuzzy environment, which also use models of the objects being optimized [59].

The flow chart of the benzene production complex in Figure 1 shows a diagram of the working process occurring at the research facility.

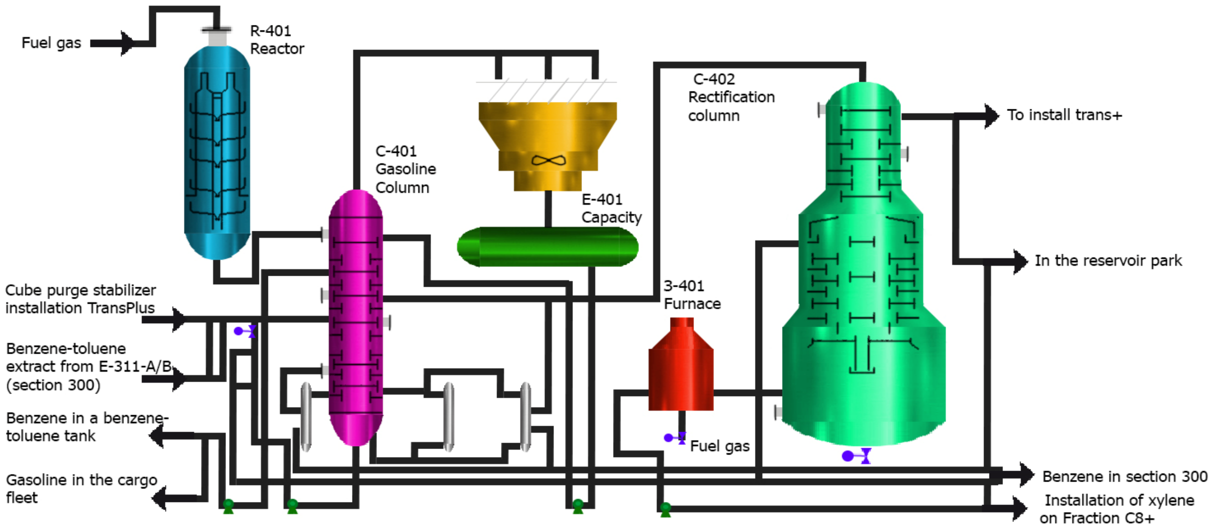

Figure 1. Working process flow chart of the benzene production complex.

Brief description of the working process of benzene production.

The raw material for obtaining benzene (fuel gas) enters the R-401 reactor, from where it enters the benzene column C-401 at a given temperature. Then, the semi-finished products are first heated to a temperature specified by the process regulations and then enter the tank of the E-401 heat exchanger; then, through an air cooler, the semi-finished product for obtaining the target product enters the C-401 column. In this C-401 column, after the appropriate processes, the final (target) product, benzene, is removed from the bottom. Some of the semi-finished products pass from the C-401 to the rectification column to obtain other products and raw materials for further processing at other facilities.

The structure of the hierarchical technological infrastructure, allowing visual analysis of the process of optimizing the operating modes of the units of the benzene production technological complex, is shown in Figure 2. The hierarchy of the optimization system consists of three levels. The first (lower) level of the system consists of sensors, controllers, and actuators. The second (middle) level of the hierarchical optimization system contains a computer system in which the models of the units of the benzene production complex, heuristic, and other optimization methods are implemented in software. At the third (upper) level of the optimization system, the operator–technologist making the decision selects the optimal operating modes when managing the benzene production complex using the interface of the optimization system.

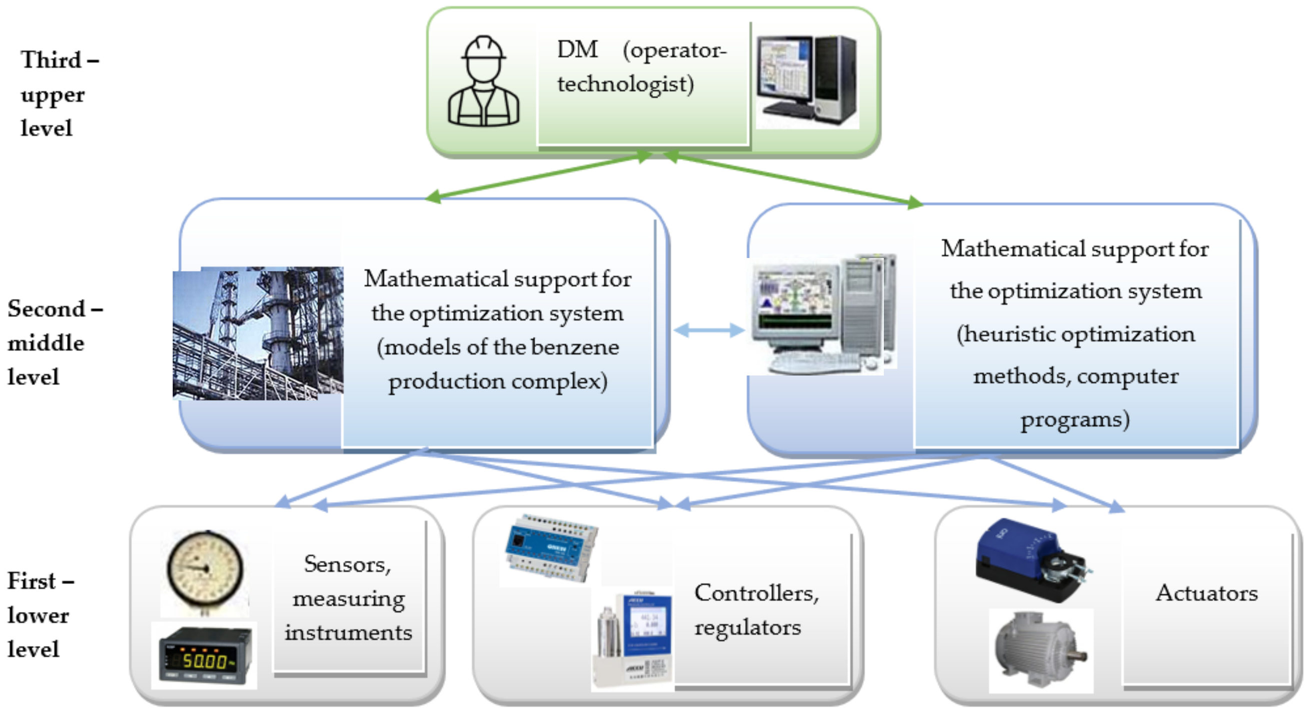

Figure 2. Hierarchical technological infrastructure of the operating modes optimization process of the benzene production complex.

The optimization strategy, according to the goals, is based on the search for optimal operating modes of the benzene production process complex, taking into account its models and the multi-criteria optimization methods proposed below. In this case, the multi-criteria optimization process consists of determining and selecting a compromise solution by the DM that ensures the efficiency of the facility's operating mode for the selected objectives, taking into account the production plan and the current situation. Potential threats (data processing, rank management, classification) are taken into account based on marketing research and the state of the benzene sales market, price changes, and consumer demands for products.

The source of fuzzy information of this study is the knowledge, experience, and preferences of the DM and experts, formalized in natural language and describing the states and operating modes of the object of study. To change their value, "linguistic variables" are used, which can convey the meaning of phrases that are complete in meaning from natural language. The mathematical definition of a linguistic variable (LV) is as follows: it represents a quintuple (tuple) $\{x,\ T(x), X, G, M\}$, where x—LV name and $T(x)$—set of values of LV x, each of which will be a fuzzy LV on X. In the provided tuple, G—a syntactic rule that forms names for new values of x, and M—a semantic procedure that transforms a new name formed by rule G into a fuzzy variable. The semantic procedure allows for the specification of the type of membership function and associates the name with its value. In addition, to build models and optimize the operating modes of the research object, theoretical information about the process and experimental and statistical data about the operating modes of the object are used.

In this study, simplex methods [1–3] and multi-criteria optimization methods [4,5,7,22,50–63] are used to develop and use multi-criteria fuzzy optimization methods. To obtain, process, and formalize fuzzy information, methods of expert assessments and fuzzy set theory are used [50–53,64–67].

The main reason for choosing and using these methods is the object of study, the operating modes of which will be optimized and which are described by a vector of criteria and are characterized by the fuzziness of some parameters and restrictions. In this regard, methods of expert assessments and theories of fuzzy sets will be used to collect, formalize, process, and use additional fuzzy information from the DM and experts.

Thus, the necessary data are collected based on various methods. The collection and processing of measurable data is based on experimental–statistical data and fuzzy data, evaluating fuzzy constraints on the quality of the manufactured product based on expert methods and theories of fuzzy sets.

The methods being developed for fuzzy optimization of the technological process under study are based on search optimization methods, namely simplex methods. In this case, it is assumed that there is an experienced DM, for example, an operator–technologist who controls the operating modes of the object and knows the ratios of the qualitative values of the optimization criteria. In practice, an experienced DM knows which indicators are more important in the current production situation and which values of fuzzy criteria are better. This justifies the need to use the knowledge, experience, and ability of the DM to effectively solve problems of multi-criteria fuzzy optimization. The proposed optimization method with a fuzzy criterion is iterative, and in each iteration of the n-dimensional problem, the DM is required to determine and select the "best" and "worst" from the set of n+*1* simplex vertices. These vertices are, respectively, the maximum and minimum values of the fuzzy criterion. Then, depending on the method used, the DM must divide the remaining vertices of the simplex into "bad", "average", and "good", based on the estimates of the quality criterion value at these vertices. To implement the proposed method, the DM is required to fuzzily estimate the values of fuzzy criteria based on his experience, knowledge, and intuition.

A. Multi-criteria optimization method with fuzzy criteria based on the simplex method.

A simplex of size n is the convex hull of $n + 1$ points (vertices of the simplex) of an affine space that does not lie in a subspace of dimension $n - 1$. By a regular n-dimensional simplex, we mean a set of $n + 1$ points equidistant from each other in an n-dimensional space.

The proposed modified and adapted simplex method for solving a multi-criteria optimization problem combines the process of studying the response surface, i.e., the dependence of the criteria vector $F(\mathbf{x}) = f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$, where $(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$—local criteria, on the parameters of the optimized process $\mathbf{x}$, in n+1-th dimensional space. In this case, the parameters being optimized are usually a vector, i.e., $\mathbf{x} = (x_1, \ldots, x_k)$.

An important property of a simplex is that by discarding several vertices of a simplex and using its remaining vertices, a new simplex can be constructed by adding several new vertices. This property of the simplex is used to develop iterative optimization methods that allow the simplex to be shifted along the response surface toward the optimum. Consider the problem of multi-criteria optimization, for example, minimizing the vector of criteria $F(\mathbf{x})$:

$$\min_{x \in E^n} F(\mathbf{x}),\ F(\mathbf{x}) = f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \tag{1}$$

where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$—measurable local criteria, and the vector $F(\mathrm{x})$ is not specified explicitly. The value of the criterion vector $F(\mathbf{x})$ based on the weight values of $f_i(\mathbf{x})$, $i = \overline{1, m}$ is compared by the DM.

Simplex Method of Multi-criteria Fuzzy Optimization (SMM-CFO).

The main points of the developed method of multi-criteria fuzzy optimization based on the simplex method are as follows:

1. With the participation of the DM, the initial vertex (point) $\mathbf{x}^1$ is determined and selected), i.e., $N = 1$;

2. A correct simplex S_N, with center $\mathbf{x}^N$, radius of the circumscribed hypersphere R_N, and vertices $\mathbf{x}^{N,j}$, $j = \overline{1, n+1}$., is constructed. Here and below, N denotes the step number, j—the number of the simplex vertex, and n—good vertices where the criterion is the best;

3. At the vertices of the simplex $\mathbf{x}^{N,j}$, $j = \overline{1, n+1}$ and in its center $\mathbf{x}^N$ with the help of the DM, the values of the local criteria $f_i(\mathbf{x}^{N,j})$, $f_i(\mathbf{x}^N)$, $j = \overline{1, n+1}$, $i = \overline{1, m}$. are measured;

4. Among the vertices of the simplex $\mathbf{x}^{N,j}, j = \overline{1, n+1}$, the DM selects its vertices with the minimum and maximum values of the vector criterion $F(\mathbf{x})$, numbered accordingly $\mathbf{x}^{N,\, n+1}$ and $\mathbf{x}^{N,1}$;

5. Vertices of the simplex $\mathbf{x}^{N,j}, j = \overline{1, n+1}$ are divided by the DM based on his knowledge and experience into m—"bad" vertices $(1 \le m \le n)$, with numbers $\mathbf{x}^{N,j},\ j = \overline{1, m}$; l—"average" vertices $(0 \le l \le n - m)$ with numbers $\mathbf{x}^{N,j},\ j = \overline{m+1, m+l}$; $n + 1 - m - l$ and n—"good" vertices with numbers $\mathbf{x}^{N,j},\ j = \overline{m+l+1, n+1}$. Here, bad vertices are those vertices of the simplex $x^{N,j},\ j = \overline{1, m},\ 1 \le m \le n$, in which the criterion values are estimated to be worse than in the other vertices of the simplex. "Good vertices" $n+1-m-l, n+1-m-l$ are such vertices of the simplex in which the values of the criteria are estimated by the DM to be better than in other vertices. "Average vertices" $x^{N,j},\ j = \overline{m+1, m+l}, n+1-m-l$ are vertices in which the values of the criteria are better than in the bad vertices but worse than in the good vertices of the simplex;

6. α = 2 is accepted, where α here and below is a parameter that determines the locations of the displayed vertices of the simplex. At 1 < α < 1, m, the displayed vertices of the simplex diverge beyond the center of the undisplayed vertices but lie closer to the hyperplane. At α = 2, the displayed vertices of the simplex are symmetric to the displayed vertices with respect to the hyperplane. At α > 2 m, displayed vertices diverge beyond the center of the undisplayed vertices and lie further from the hyperplane than the corresponding displayed vertices [2];

7. Based on the standard simplex method, the simplex S_{N+1} is constructed using the following formulas:

$$\mathbf{x}^{N+1,j} = (\alpha - 1)\mathbf{x}^{N,j} + \frac{2-\alpha}{n+1-m-l}$$

$$\sum_{i=m+l+1}^{n+1} x^{N,i},\ j = \overline{1, n+1},$$

$$\mathbf{x}^{N+1} = (\alpha - 1)\mathbf{x}^{N} + \frac{2-\alpha}{n+1-m-l} \sum_{i=m+l+1}^{n+1} \mathbf{x}^{N,i}.$$

7.1. If $\alpha < 1$, then go to step 8; if $\alpha < 1$, otherwise, i.e., then $\alpha \ge 1$, the transition is made to the next step 7.2;

7.2. Using the idea of the standard simplex method, the following simplex S_{N+1} is constructed using the following formulas:

$$\mathbf{x}^{N+1,j} = \mathbf{x}^{N+1,j} + 2\Delta_{N+1}(m,l),\ j = \overline{1, m},$$

$$\mathbf{x}^{N+1,j} = \mathbf{x}^{N+1,j} + \frac{2m\Delta_{N+1}(m,l)}{(n+1-l)}\ j = \overline{m+1, m+l},$$

$$\mathbf{x}^{N+1,j} = \mathbf{x}^{N+1,j},\ j = \overline{m+l+, n+l},$$

$$\mathbf{x}^{N+1} = \mathbf{x}^{N+1} + \frac{2m\Delta_{N+1}(m,l)}{(n+1-l)}\ j = \overline{m+1, m+l},$$

$$\Delta_{N+1}(m,l) = \sum_{i=m+l+1}^{n+1} \frac{\mathbf{x}^{N+1,j}}{(n+1-m-1)} - \sum_{i=1}^{m} \frac{\mathbf{x}^{N+1,j}}{m}.$$

In these expressions above, Δ_{N+1} denotes the change in the "bad" and "average" vertices determined by the DM when displaying simplices.

Based on the above formulas, the next simplex closer to the optimum is constructed by displaying bad vertices into better ones, which improves the solution.

In practice, the decision to terminate and stop the search is often assigned to DMs, who, in conditions of multi-criteria and fuzziness, can determine the best solutions. The condition for stopping the search process can be $N \ge N_0$ or $R_N \le R'$, where R_N—simplex size, N_0, and R'—given numbers;

8. The fulfillment of the stopping condition for searching for the optimal solution is checked. If the rule is not satisfied, then go to step 9. If the condition for stopping the search is satisfied, then the search stops, and the vertex of the simplex with the best value of the optimized criterion vector is remembered;

9. At the center of the simplex $\mathbf{x}^{N+1}$, the DM measures the values of local criteria $f_i\left(\mathbf{x}^{N+1}\right),\ i = \overline{1,m}$;

10. The DM compares the values of the criteria $F\left(\mathbf{x}^N\right)$ and $F\left(\mathbf{x}^{N+1}\right)$. If $F\left(\mathbf{x}^{N+1}\right) < F\left(\mathbf{x}^N\right)$, then go to step 13. Note that here, $F\left(\mathbf{x}^N\right)$ is the vector of criteria at the center of the simplex, estimated by the DM, for example, by the Pareto method, taking into account the weight coefficients of local criteria specified by the DM;

11. At the vertices of the simplex $f_i\left(\mathbf{x}^{N+1,j}\right)$, not coinciding with the vertices $f_i\left(\mathbf{x}^{N,j}\right)$, $j = \overline{1, n+1}$, the DM performs measurements and evaluation of local criteria values $f_i\left(\mathbf{x}^{N+1,j}\right),\ j = \overline{1, n+1},\ i = \overline{1,m}$;

12. Similar to standard simplex methods, the transition is made to the next step of searching for the optimum: $N = N + 1$ and return to step 4 to start the next optimization iteration;

13. If $\alpha = 2$, then $\alpha = 1.5$ and go to step 7; if $\alpha = 1.5$, then $\alpha = 0.5$ and go to step 7;

If $\alpha < 1$, then $\alpha = \alpha = \frac{\alpha}{2}$ and go to step 7.

Thus, the proposed modified and adapted simplex method of multi-criteria fuzzy optimization is based on the knowledge and experience of the DM and experts, which can be expressed fuzzily, for the formalization of the methods of fuzzy set theories and the Pareto principle of optimality are used. In fact, due to these methods and the principle, a multi-criteria fuzzy problem is formalized and solved as a single-criterion problem with formalized data based on the methods of fuzzy sets, knowledge, experience, and intuition of the DM and experts.

B. Heuristic method for multi-criteria optimization in a fuzzy environment.

A multi-criteria optimization problem in a fuzzy environment can be formulated as a fuzzy mathematical programming problem as follows [6,7]:

$$\max_{\mathbf{x}\in X} \mu_C^i(\mathbf{x}), i = \overline{1,m}, \tag{2}$$

$$X = \left\{ \mathbf{x} : \arg\max_{\mathbf{x}\in\Omega} \mu_q(\mathbf{x}),\ q = \overline{1,L} \right\}, i = \overline{1,m}, \tag{3}$$

where $\mu_C^i(\mathbf{x}), i = \overline{1,m}$—membership functions of fuzzy local criteria, and if they are crisp, then the normalized values of the criteria that take values in the interval [0, 1]; $\mathbf{x} = (x_1, \ldots, x_k)$—vector of independent variables (input, operating parameters of the object), changing that ensures optimal values of the criteria; X—area of feasible solutions, taking into account fuzzy constraints; Ω—the initial set of alternatives, in production, is determined by the facility's operating regulations; and $\mu_q(x),\ q = \overline{1,L}$—membership functions describing the degree of fulfillment of fuzzy constraints.

To clarify and obtain a correct formulation of the multi-criteria optimization problem (2)–(3), it is necessary to use various optimality principles modified for working in a fuzzy environment. For example, by modifying *the principles of maximin and Pareto optimality*, the multi-criteria optimization problem in a fuzzy environment can be written as follows [59,63]:

$$\max_{x\in X} \mu_C^1(\mathbf{x}), \tag{4}$$

$$X = \left\{ \begin{array}{l} \mathbf{x} : arg\max_{\mathbf{x}\in\Omega} \min_{i\in I_C} \left(\gamma_i \mu_C^i(\mathbf{x})\right) \wedge arg\max_{x\in\Omega} \sum_{q=1}^{L} \beta_q \mu_q(\mathbf{x}) \wedge \\ \sum_{q=1}^{L} \beta_q = 1 \wedge \beta_q \geq 0,\ I_C = \{2, \ldots, m,\ q = \overline{1,L}\} \end{array} \right\} \tag{5}$$

where $\wedge$—a logical "and" sign that requires the truth of all expressions connected through it. Other designations are described above.

In the formulated statement of the problem of multi-criteria fuzzy optimization, the multi-criteria problem is transformed into a single-criterion problem by optimizing the most important criterion, taking into account the fuzziness of the constraints. The remaining criteria are taken into account in the composition of constraints based on the modification of the maximin principle, and the fuzziness of the criteria is taken into account based on the modification of the Pareto optimality principle. The details of this idea are disclosed in the proposed and described below heuristic method for solving problems (4)–(5) based on the modification and combination of the maximin and Pareto optimality principles.

In the resulting formulation of the multi-criteria optimization problem in a fuzzy environment (4)–(5), the main, most important criterion $\mu_C^1(\mathbf{x})$ is maximized, and the remaining local criteria $\mu_C^i(\mathbf{x}),\ i=\overline{2,m}$, taking into account their weighting coefficients, are taken into account as part of the constraints. In this case, the most important criterion $\mu_C^1(\mathbf{x})$ and the weight coefficients of local criteria γ_i are determined by the DM, and local criteria with weight coefficients are included in the constraints based on the modified maximin principle. The fulfillment of the requirements of fuzzy constraints is taken into account through their membership functions $\mu_q(\mathbf{x}),\ q=\overline{1,L}$ based on the Pareto principle of optimality.

In the formulated statement of the problem of multi-criteria fuzzy optimization, the multi-criteria problem is transformed into a single-criterion problem by optimizing the most important criterion, taking into account the fuzziness of the constraints.

Heuristic method of multi-criteria optimization based on modification and combination of maximin and Pareto optimality principles.

1. The DM selects the most important optimizable criterion $\mu_C^1(\mathbf{x})$, and the vector of weight coefficients $\gamma=(\gamma_2,\ldots,\gamma_m)$ is specified for other local criteria $\mu_C^i(\mathbf{x}),\ i=\overline{2,m}$, taken into account as part of the constraints. In this case, it is set in compliance with the following requirements:

$$\sum_{i=2}^{m}\gamma_i=1,\ \gamma_i\geq 0,\ i=\overline{2,m},$$

2. If criteria $\mu_C^1(\mathbf{x}),\ \mu_C^i(\mathbf{x}),\ i=\overline{2,m}$, the weighting coefficients $\gamma_2,\ldots,\gamma_m$ are fuzzy, then for them, with the participation of DMs and experts, term sets are determined, and membership functions are constructed;

3. With the involvement of the DM and experts, the values of the vector of weighting coefficients of the importance of fuzzy constraints are determined and entered $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_L)$, $\sum\limits_{q=1}^{L}\beta_q=1,\ \beta_q\geq 0,\ q=\overline{1,L}$;

4. For each q-th coordinate of the constraints, the number of steps is specified p_q, $q=\overline{1,L}$.

5. Using the formula $h_q=\frac{1}{p_q}$, the size of steps is calculated to change the coordinates of the vector of weighting coefficients $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_L)$;

6. Changing in the interval [0, 1] with step h_q, a set of weight vectors is constructed $\beta^1,\ \beta^2,\ldots,\beta^N$, $N=(p_1+1)\cdot(p_2+1)\cdot\ldots\cdot(p_L+1)$;

7. To describe fuzzy constraints, a term set is determined, and membership functions are constructed $\mu_q(\mathbf{x}),\ q=\overline{1,L}$, assessing the degree of fulfillment of fuzzy constraints;

8. Based on mathematical models of the object being optimized, describing the dependence of criteria and restrictions on x, the main criterion $\mu_C^1(\mathbf{x})$ (4) is maximized on the set of feasible solutions X, determined by the expression (5). The current solutions are determined: $\mathbf{x}(\boldsymbol{\gamma},\boldsymbol{\beta})$—the value of independent variables (input, operating parameters), providing the corresponding values of the criteria $\mu_C^1(\mathbf{x}),\ \mu_C^i(\mathbf{x}),\ i=\overline{2,m}$ and the degrees of fulfillment of fuzzy constraints $\mu_q(\mathbf{x}),\ q=\overline{1,L}$;

9. Current decisions obtained in the previous paragraph are presented to the DM for analysis, adjustment, or making a final decision. If the current solutions do not satisfy the DM, then in order to improve the solution, the values of $\boldsymbol{\gamma}$ and/or $\boldsymbol{\beta}$ are adjusted, and a

return is made to step 4 to find the best solution. If the current decisions satisfy the DM, the final, best decision is made taking into account the current situation in production;

10. The search for a solution stops, and the final solution adopted by the DM is displayed: $\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta})$, providing the maximum value of the main criterion $\mu_C^1(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta}))$; guaranteed values of other local criteria $\mu_C^i(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta}))$, $i = \overline{2, m}$; and maximum degrees of the fulfillment of fuzzy constraints $\mu_q(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta}))$, $q = \overline{1, L}$.

4. Results

Let us present the results to determine the convergence of the proposed simplex method. To determine the convergence of the SMMFO method and estimate its convergence rate, set the class of criteria to be optimized and, more specifically, determine the conditions for improving the criterion values at the centers of simplexes. Let us assume that the condition is satisfied:

$$F\left(\mathbf{x}^{N+1}\right) - F\left(\mathbf{x}^{N}\right) \leq \varepsilon \left\| \mathbf{x}^{N+1} - \mathbf{x}^{N} \right\|^2,$$

where ε—a small constant that is positive.

Then, taking this assumption into account, we can formulate the following theorem on the convergence of the SMM-CFO method.

Theorem 1. *Let the optimized criteria vector $F(\boldsymbol{x})$ be convex and satisfy the following conditions:*

1. $\|gradF(\mathbf{x}) - gradF(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \mathbf{x}, \mathbf{y} \in E^n,\ L - const;$

2. Lebesgue set $M(\mathbf{x}) = \left\{\mathbf{x} : F(\mathbf{x}) \leq F\left(\mathbf{x}^1\right)\right\}$ is restricted.

Then, the sequence of simplex vertices $\left\{\mathbf{x}^N\right\}$,generated by the proposed simplex method, is minimized, and the limiting value of the criterion vector $F_{min}(\boldsymbol{x})$ will be the best. In this case, the estimate will be valid:

$$F\left(\mathbf{x}^N\right) - F_{min}(\mathbf{x}) \leq \frac{F(\mathbf{x}^1) - F_{min}(\mathbf{x})}{1 + (F(\mathbf{x}^1) - F_{min}(\mathbf{x})C(N-1)/H_1},\ N = 1, 2, \ldots \tag{6}$$

where $f_{min} = \min\limits_{\mathbf{x} \in E^n} F(\mathbf{x}),\ H_1 = diamM(\mathbf{x}),$
for the SMM-CFO method

$$C = \frac{\varepsilon \alpha_0^2}{L^2 n^4} \left(\frac{1 - \sqrt{1 - cos^2\varphi_{m,l}}}{1 - \sqrt{1 + cos^2\varphi_{m,l}}} \right)^2,$$

where $\alpha_0 > 0$—the initial value of the parameter α, which determines the locations of the displayed vertices of the simplex; and $cos^2\varphi_{m,l}$—cosine of the angle of the displayed bad and average values.

Proof of Theorem 1. Let us show that in the attached SMM-CFO method, it is always possible to select parameters that ensure the fulfillment of the conditions:

$$-\left(P_N^i,\ gradF\left(\mathbf{x}^N\right)\right) = \left\| gradF\left(\mathbf{x}^N\right) \right\| cos\Theta_N > 0, \tag{7}$$

$$\Phi_1 > \Phi_2 > \ldots,\ \Phi_j = F\left(\mathbf{x}^j\right) - F_{min}(\mathbf{x}), \tag{8}$$

where $\Phi_1, \ldots, \Phi_j$—sequence of vertex numbers of constructed simplices.

Then, according to the Polyak–Tsypkin theorem [68], $\left\{\mathbf{x}^N\right\}$ will be a minimizing sequence. □

The value of the angle Θ_N is determined by the error in determining the gradient direction based on the estimates of the criterion $F(\boldsymbol{x})$ at the vertices of the simplex and the solution to the problem of choosing the directions of displacement of the center of the simplex. The estimate of the gradient vector based on measurements at the vertices of the simplex can be represented as follows [62]:

$$gr\hat{a}dF\left(x^N\right) = gradF\left(\mathbf{x}^N\right) + \frac{n}{(n+1)^2}\sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\left(\frac{A_{ji}}{R_N} + \left(G_i, r^{N,j}\right)\right)r^{N,j},$$

where $A_{ji} = F(\mathbf{x}^{N,j}) - F(\mathbf{x}^{N,i}) - R_N(gradF(\mathbf{x}^{N,i}), r^{N,j}), G_i = gradF(\mathbf{x}^{N,i}) - gradF(\mathbf{x}^N)$.

Then, the following is the estimate for the cosine of the angle β_N between $gr\hat{a}dF(\mathbf{x}^N)$ and $gradF(\mathbf{x}^N)$:

$$cos\beta_N \geq \frac{\| gradF(\mathbf{x}^N)\| - 2LnR_N}{\| gradF(\mathbf{x}^N)\| + 2LnR_N}. \tag{9}$$

The angle $\varphi_N^{(i)}$ is the angle between the direction of displacement of the simplex center and the estimate $gr\hat{a}dF(\mathbf{x}^N)$. The above condition (7) will be satisfied if

$$cos\Theta_N \geq cos\left(\beta_N + \varphi_N^{(i)}\right) > 0. \tag{10}$$

Substituting estimate (9) into (10) and making the necessary transformations, we find that condition (7) is satisfied for

$$R_N = \frac{\| gradF(\mathbf{x}^N)\|}{2Ln} \tag{11}$$

Taking into account the fact that for convex criteria

$$F\left(\mathbf{x}^N\right) - F\left(\mathbf{x}^{N+1}\right) \geq \left(gradF\left(\mathbf{x}^N\right), \mathbf{x}^N - \mathbf{x}^{N+1}\right) - \frac{L}{2}\left(\mathbf{x}^{N,i}\right) - \left\|\mathbf{x}^{N+1} - \mathbf{x}^N\right\|^2.$$

it can be shown that in the case when the rules of simplex methods $F^*(\mathbf{x}^N) = F(\mathbf{x}^N)$, at

$$R_N \leq \frac{n\| gradF(\mathbf{x}^N)\| cos\Theta_N}{2\varepsilon + L},$$

and when

$$F^*\left(x^N\right) = \frac{1}{n+1}\sum_{i=1}^{n+1} F\left(\mathbf{x}^{N,i}\right),$$

at

$$R_N \leq \frac{4n\| gradF(\mathbf{x}^N)\| cos\Theta_N}{8\varepsilon + 5L},$$

condition (9) is satisfied.

From the last condition, we note that by choosing the required size of the simplex R_N, it is possible to satisfy the convergence conditions (7), (8). It should be noted that during the search, conditions (7), (8) will be fulfilled since the structure of the simplex method provides for changing the size of the simplex in accordance with the rule $R_N = \gamma(t)R_1$, where $\gamma(t)$—simplex parameter, which is a positive value $\gamma(t) \leq 1$, t—simplex compression number, and R_1—the size of the first simplex and equal to the radius of the circumscribed hypersphere with the center at $\mathbf{x}^1$.

In order to obtain convergence estimates, we use the following condition [69]:

$$\Phi_N \leq \Phi_1 / \left(1 + \frac{\Phi_1}{H_1}\sum_{i=1}^{N-1}\frac{F(\mathbf{x}^i) - F(\mathbf{x}^{i+1})}{\|gradF(\mathbf{x}^i)\|^2}\right). \tag{12}$$

It is known that for convex functions (criteria), the following inequality is true:

$$F^*\left(\mathbf{x}^N\right) - F\left(\mathbf{x}^N\right) \geq 0, F\left(\mathbf{x}^{N+1}\right) - F^*\left(\mathbf{x}^{N+1}\right) \geq -R_N L/2.$$

Here, $F^*\left(\mathbf{x}^N\right)$ is determined by the following formula:

$$F\left(\mathbf{x}^N\right) = \frac{1}{n+1}\sum_{i=1}^{n+1} F\left(\mathbf{x}^{N+i}\right). \quad (13)$$

Then, from the geometric rules for constructing a new simplex, the following inequality holds: $\left\|\mathbf{x}^N - \mathbf{x}^{N+1}\right\| \geq 2R_N/n$. Based on this inequality and the condition for violation of which the size of the simplex changes $F^*\left(\mathbf{x}^N\right) - F^*\left(\mathbf{x}^{N+1}\right) \leq \varepsilon\left\|\mathbf{x}^N - \mathbf{x}^{N+1}\right\|^2$, we find that for the simplex method with $F^*\left(\mathbf{x}^N\right) = F\left(\mathbf{x}^N\right)$, the following condition is true:

$$F\left(\mathbf{x}^N\right) - F\left(\mathbf{x}^{N+1}\right) \geq 4\varepsilon R_N^2/n^2. \quad (14)$$

For the simplex method with $F^*\left(\mathbf{x}^N\right)$, determined by formula (13)

$$F\left(\mathbf{x}^N\right) - F\left(\mathbf{x}^{N+1}\right) \geq \left(\frac{4\varepsilon}{n^2} - \frac{L}{2}\right)R_N^2. \quad (15)$$

From the well-known rules of simplex methods [62] and inequality (11), it follows that there is such a number $\alpha_0 \geq 0$, and that the following condition is satisfied:

$$R_N \geq \frac{\alpha_0\left\| gradF\left(\mathbf{x}^N\right)\right\|}{2Ln}. \quad (16)$$

Based on formulas (12)–(16), we can conclude that estimate (6) of the proposed systemic SMM-CFO method is valid. The theorem is proven.

Using the results provided in the study [62] for the proposed simplex method SMM-CFO for $n \geq 2$, it can be determined that

$$cos\varphi_{m,l}^N \geq \frac{2n^2+1}{(n+1)\sqrt{n(5n+4)}} = cos\varphi_{m,l}. \quad (17)$$

The provided estimate (17) shows that with the increase in the size of the optimization problems, the convergence rate of the simplex method SMM-CFO increases, where m, l, and n are described above in the fifth step of the proposed simplex method.

Next, we will consider the results of applying the heuristic method of multi-criteria fuzzy optimization proposed in the previous section to solve a real production problem in the production of benzene.

First, we formalize the problem of multi-criteria optimization of the benzene production process to specify and correctly formulate this multi-criteria optimization problem in a fuzzy environment.

Let $\mu_C(\mathbf{x}) = \left(\mu_C^1(\mathbf{x}), \mu_C^2(\mathbf{x}), \mu_C^3(\mathbf{x})\right)$ be the normalized vector of criteria that determines the yield of products from the benzene production process complex based on measurements, where $\mu_C^1(\mathbf{x})$—volume of benzene from benzene column; $\mu_C^2(\mathbf{x})$—volume of raffinate from a given column; and $\mu_C^3(\mathbf{x})$—volume of heavy aromatics from the distillation column. In this case, the normalization of the values of these criteria can be performed using the expression $\frac{\mu_C^i(\mathbf{x})}{\max\mu_C^i(\mathbf{x})}, i = \overline{1,3}$, i.e., by dividing their values by the possible maximum value, where $\mu_C^i(\mathbf{x}), i = \overline{1,3}$—current values of the criteria obtained by measurement; and $\max\mu_C^i, i = \overline{1,3}$—the maximum value of these criteria, the value of which is known in practice.

Supposing they were constructed by the DM, membership functions were constructed by experts $\mu_q(\mathbf{x}),\ q = \overline{1,2}$, determining the degree of fulfillment of fuzzy constraints: “the average octane number of benzene is not lower, i.e., $\tilde{\geq}$ 102»; “the sulfur content in benzene is no more, i.e., $\tilde{\geq}$ 0.00005%». We also believe that the values of the vectors of weighting coefficients of the importance of the criteria $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ and constraints $\boldsymbol{\beta} = (\beta_1, \beta_2)$ have been determined

Then, the problem of multi-criteria optimization of the benzene production process in a fuzzy environment based on the general formulation (4)–(5) can be written in the following form:

$$\max_{\mathbf{x}\in X} \mu_C^1(\mathbf{x}), \tag{18}$$

$$X = \left\{ \begin{array}{l} x : arg\max_{\mathbf{x}\in\Omega} \min_{i\in I_C} \left(\gamma_i \mu_C^i(\mathbf{x})\right) \wedge arg\max_{x\in\Omega} \sum_{q=1}^{2} \beta_q \mu_q(\mathbf{x}) \wedge \\ \sum_{q=1}^{2} \beta_q = 1 \wedge \beta_q \geq 0,\, I_c = \{2,\, 3\},\, q = 1, 2. \end{array} \right\} \tag{19}$$

The solution to the problem of multi-criteria optimization in a fuzzy environment (18)–(19) consists of finding and determining such values of independent variables $x^*(\gamma, \beta)$ that maximize the values of the most important criterion $\mu_C^1(\mathbf{x})$. In this case, according to the formulation of the problem (18)–(19), the values of the remaining two local criteria $\mu_C^i(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta}))$, $i = \overline{2,3}$ must be ensured according to the maximin principle, and the conditions of fuzzy constraints $\mu_q(\mathbf{x})$, $q = 1, 2.$ must not be violated.

The problem of multi-criteria optimization of the benzene production process in a fuzzy environment (18)–(19) was solved based on the heuristic MM + PO method proposed in Section 2. Let us present the results of solving the problem of multi-criteria optimization of the benzene production process.

1. The DM chose $\mu_C^1(x)$—the volume of benzene from the benzene column as the most important optimized criterion and set the vector of weighting coefficients $\boldsymbol{\gamma} = (0.6.0.3, 0.1)$, 0.6 + 0.3 + 0.1 = 1 for the remaining local criteria $\mu_C^i(\mathbf{x})$, $i = 2, 3$, which are taken into account as part of the constraints;

2. Since, in our case, $\mu_C^1(\mathbf{x})$, $\mu_C^i(\mathbf{x})$, $i = 2, 3$ and $\gamma_1, \gamma_2, \gamma_3$ are crisp, for them, the term set is not defined, and membership functions are not constructed. The criteria are normalized to take values in the range [0, 1] and are determined based on models of the benzene and distillation column of the benzene production complex developed with the participation of the authors in the work [70];

3. With the involvement of the DM and experts, the values of the vector of weighting coefficients of the importance of fuzzy constraints were determined and entered $\boldsymbol{\beta} = (0.6,\ 0.4)$, from the condition 0.6 + 0.4 = 1;

4. For each q = th coordinate of the constraints, the number of steps is specified $p_1 = 4$, $p_2 = 2$;

5. Using the formula $h_q = \frac{1}{p_q}$, $q = 1, 2$, the size of steps was calculated to change the coordinates of the vector of weighting coefficients $\boldsymbol{\beta} = (0.6,\ 0.4) : h_1 = \frac{1}{p_1} = \frac{1}{4} = 0.25$; $h_2 = \frac{1}{p_2} = \frac{1}{2} = 0.5$;

6. Changing in the interval [0, 1] with steps h_q, $q = 1, 2$, a set of weight vectors is constructed β^1, $\beta^2, \ldots, \beta^{15}$, $N = (4+1)\cdot(2+1) = 15$;

7. To describe the fuzzy constraints, "average octane number of benzene $\tilde{\geq}$ 102", "sulfur content in benzene $\tilde{\geq}$ 0.0000" a term set {low, below average, average, above average, high] is defined, and membership functions $\mu_q(\mathbf{x})$, $q = 1, 2$, are constructed for them to assess the degree of fulfillment of fuzzy constraints:

$$\mu_1^4(\mathbf{x}) = \exp(0.5|y_4 - 106|)^{0.55};\ \mu_1^5(\mathbf{x}) = \exp(0.5|y_4 - 108|)^{0.58};$$

$$\mu_2^1(\mathbf{x}) = \exp(0.4|y_5 - 0.00002|)^{0.14};\ \mu_2^2(\mathbf{x}) = \exp(0.4|y_5 - 0.00003|)^{0.12};$$

$$\mu_2^3(\mathbf{x}) = \exp(0.4|y_5 - 0.00005|)^{0.10};\ \mu_2^4(\mathbf{x}) = \exp(0.4|y_5 - 0.00006|)^{0.12};$$

$$\mu_2^5(\mathbf{x}) = \exp(0.4|y_5 - 0.00007|)^{0.14},$$

where $\mu_1^t(\mathbf{x}),\ \mu_2^t(\mathbf{x}),\ t = \overline{1,5}$—membership functions estimating the degree of fulfillment of fuzzy constraints on the quality of benzene, i.e., on the average octane number of benzene and on the sulfur content in the composition of benzene for each term; t—numbers of terms in the term set; y_4, y_5—numerical values of the membership function determining the qualities of benzene at α levels; 0.5, 0.4—coefficients of rough adjustment of the form of the membership function, determined during the parametric identification of the membership function with a value of α-level equal to 0.5; and 0.58, 0.55, 0.52, 0.14, 0.12, 0.10—coefficients for fine-tuning the shape of the membership function;

8. Based on the models of optimized objects that we have developed [70], which describe the dependence of criteria and constraints on the vector of their input parameters, the main criterion $\mu_C^1(x)$ (18) is maximized on the set of feasible solutions X, defined by expression (19). Current solutions are identified: $\mathbf{x}(\boldsymbol{\gamma}, \boldsymbol{\beta})$—the value of the input, operating parameters, providing the corresponding values of the criteria $\mu_C^1(\mathbf{x}),\ \mu_C^i(\mathbf{x}),\ i = 2, 3$ and degrees of fulfillment of fuzzy constraints $\mu_1(\mathbf{x})$ and $\mu_2(\mathbf{x})$. Given the resulting problem, a suitable single-criterion optimization algorithm with constraints can be used since fuzzy constraints are formalized and represented through their membership functions. We used a penalty function algorithm;

9. The current decisions obtained at the previous point are presented to the DM for analysis, adjustment, or making a final decision. In the first four iterations, the DM was not satisfied with the current solutions, and to improve the solution, he adjusted the values of $\gamma\ \beta$ and returned to step 4 to search and select the best solution. The solutions obtained at the fifth iteration with $\gamma = (0.75.0.25, 0.1)$ and $\beta = (0.77,\ 0.23)$, satisfied the DM, and taking into account the current situation and preference, he makes the final, best decision, and the transition is made to the next step 10;

10. The search for a solution is stopped, and the final solution selected by the DM is displayed: $\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta})$, providing the maximum value of the main criterion $\mu_C^1(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta}))$; guaranteed values of other local criteria $\mu_C^i(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta})),\ i = 2, 3$; and maximum degrees of fulfillment of fuzzy constraints $\mu_q(\mathbf{x}^*(\boldsymbol{\gamma}, \boldsymbol{\beta})),\ q = 1.2$. These results are listed in the third column of Table 1.

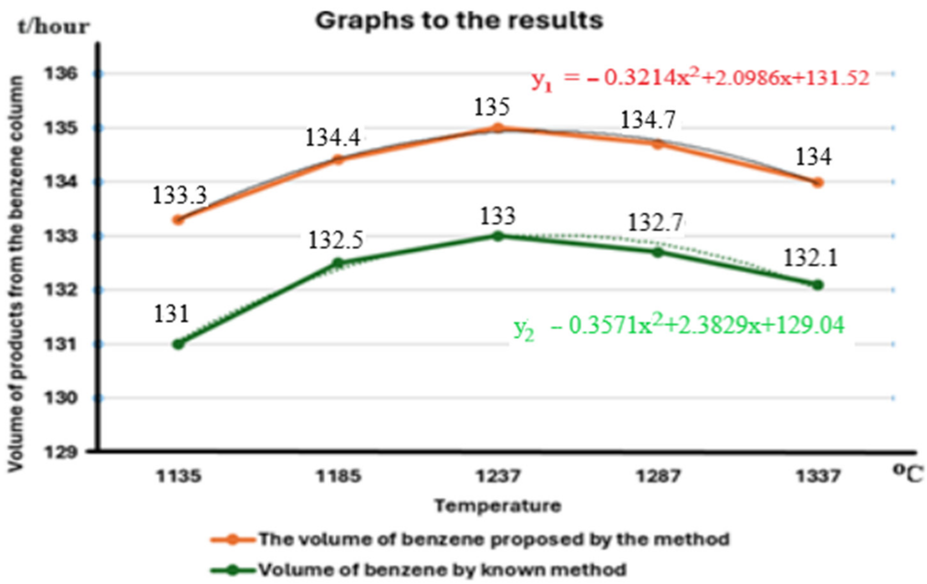

Figure 3. Graphs of results for benzene volume. The green dotted and black lines in the figure mean smoothing of the benzene volume dependencies on temperature according to the proposed and known methods.

Table 1. Comparison of results of optimization benzene production process based on a known deterministic method [71], proposed heuristic method, and real data obtained on the research object when optimizing its work experimentally.

Criteria and Constraints and Optimal Values of Input and Operating Parameters	Known Method [71]	Proposed Heuristic Method (MM + PO)	Real Data
$\mu_C^1(\mathbf{x})$—volume of benzene from benzene column, t/hour	133	135	132.8
$\mu_C^2(\mathbf{x})$—volume of raffinate from benzene column, t/hour	82	82.7	82.3
$\mu_C^3(\mathbf{x})$—volume of heavy aromatics from the distillation column, t/hour	450	450	450
$\mu_1(\mathbf{x})$—membership function of fulfilling the fuzzy constraint "average octane number of benzene $\tilde{\geq}$ 102"	–	1.0	$(\cdot)^L$
$\mu_1(\mathbf{x})$—membership function of fulfilling the fuzzy constraint "sulfur content in benzene $\tilde{\geq}$ 0.00005%"	–	1.0	$(\cdot)^L$
Optimal values of input and operating parameters of a benzene production complex: $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$: x_1^*—consumption of raw materials (reformate), thousand tons	746	746	746
x_2^*—temperature in benzene column, °C	1240	1237	1242
x_3^*—pressure in benzene column, kg/cm^2	37	34	35
x_4^*—share of sulfur in the raw material composition, %	0.005	0.005	0.005
x_5^*—the share of aromatic hydrocarbons in the composition of raw materials, %	50	50	50

Note: – means that the corresponding parameters are not determined by this method; $(\cdot)^L$: these parameters are not measured but are assessed in the laboratory with the participation of specialists.

Based on the results obtained, provided in Table 1, we will draw up graphs that allow visual comparison of the obtained results. Figures 3 and 4 show comparative graphs of the results obtained based on the well-known deterministic method from source [71] (y_2, in green) and the proposed heuristic method (y_1, in red) for the volumes of benzene (target product) and raffinate.

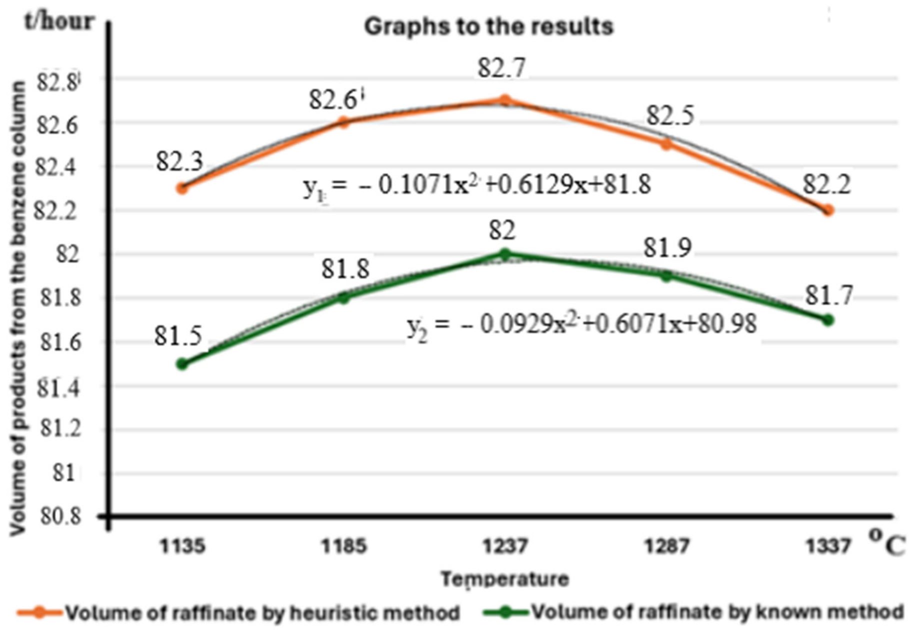

Figure 4. Graphs of results for volume of raffinate. The dotted and black lines, respectively, smooth the dependence of the raffinate volume on temperature according to the proposed method and the known method.

Based on the research conducted to assess the sensitivity analysis of the impact of initial parameters on the volume and quality of products, the following can be noted:

- Initial parameters—the values of the input and operating parameters of the benzene column affect the results, i.e., the volumes and quality indicators non-linearly (see Figures 3 and 4 and Table 1);
- The maximum influence of input and operating parameters on the obtained products is observed in their optimal values, which provide extreme values of the optimization criteria, for example, $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$, provided in Table 1.

The essence of the fuzzy rank classifier scheme of the object operating modes depending on the production situation when managing an integrated system is as follows. The DM, using a computer program for implementing models and heuristic methods (Figure 2), optimizes the modes of the integrated benzene production system in a dialogue box, the process flow chart of which is shown and described in Figure 2. In this case, multi-criteria optimization in a fuzzy environment is carried out, taking into account the rank classifier of the object operating modes depending on the production situation and production plan and taking into account changes in demand and requirements for the manufactured products (benzene). To take these fuzzy factors into account, the DM uses his experience, knowledge, intuition, and preferences depending on the rank of the situation that has arisen, as well as the results of marketing research on the state of the product market and the production plan, expressed in natural language.

5. Discussion

The method of multi-criteria optimization with fuzzy criteria proposed in Section 2 is based on the application of the idea of the simplex method and the involvement of the DM to compare the values of the criteria vector at the vertices of the simplex based on their knowledge and experience. The theorem formulated and proven in Section 3 substantiates that the sequence $\{\mathbf{x}^N\}$, generated by the proposed simplex method, is a minimizing one, and any limit point will be a minimum point $f(\mathbf{x})$. The formulated lemma shows that as the size of the optimization problem increases, the convergence rate of the proposed simplex method SMM-CFO increases.

A heuristic method of multi-criteria optimization in a fuzzy environment, developed on the basis of a modification of the maximin and Pareto optimality methods, is modified for fuzziness by normalizing the criteria and constructing a membership function for fulfilling fuzzy constraints. In this case, the idea of the maximin method is used for the criterion to present a multi-channel problem as a single-criterion one, and the idea of Pareto optimality is used to formalize and take into account the vector of fuzzy constraints based on their membership function.

The developed heuristic method for solving the problem of multi-criteria optimization of a problem in a fuzzy environment is iterative and is based on the involvement of the DM and the capabilities of computer technology in the process of solving the problem. The method is implemented in the form of a dialogue procedure between the DM and computers and allows the use of the capabilities and advantages of humans and computers. In this case, the DM, based on his knowledge and experience, solves the non-formalized part of the problem, analyzes, and selects the final solution, which is the best in the current production situation, and the formalized part of the problem is solved by a computer.

The practical application of the developed heuristic method MM + PO for solving a multi-criteria optimization problem with fuzzy constraints shows its effectiveness and advantages under fuzzy conditions compared to known methods. Since qualitative indicators (average octane number, the proportion of sulfur in benzene) are not directly measured but are assessed with human participation in laboratory conditions, they are characterized by fuzziness. These qualitative indicators of benzene are taken into account in the problem being solved in the form of fuzzy criteria.

Analyzing and discussing the obtained results of optimization of the benzene production process based on the well-known deterministic method, the proposed heuristic method, and real data obtained from experiments provided in Table 1, we can highlight the following advantages of the proposed heuristic method:

1. The values of more important criteria: those assessing the volume of benzene and raffinate, having importance coefficients of 0.75 and 0.25, respectively, have been improved. As can be seen from the data in Table 1, the volume of benzene produced, which is the most important product, increased by 2.0 t/hour or 1.5% per hour. If we take into account that this production is continuous, then 48 tons more benzene is produced per day, or its volume increases by 35%, which ensures significant production efficiency. In addition, the heuristic method allows the increase in the volume of raffinate by 0.8 t/hour or 0.98% per hour. At the same time, the yield of heavy aromatics in the compared methods is the same. From these results, we can conclude that the volume of more important products increases, and the waste part of production decreases. And this allows for improvement in the environmental condition of production;

2. The proposed heuristic method for solving a multi-criteria problem with fuzzy constraints allows us to estimate the degree of fulfillment of fuzzy constraints through their membership functions $\mu_1(\mathbf{x}),\ \mu_2(\mathbf{x})$), which are not determined by other methods. This allows us to effectively and correctly solve the optimization problem in a fuzzy environment;

3. The proposed heuristic approach to solving multi-criteria fuzzy optimization problems allows us to obtain better solutions at lower temperatures and pressures, which ensures the effectiveness of the approach.

In addition, when solving the problem of multi-criteria optimization of the operating modes of the object under study in a fuzzy environment based on the proposed heuristic method and modification of the principles of maximin and Pareto optimality, the efficiency of solving production problems characterized by fuzziness increases.

The proposed approach to solving multi-criteria optimization problems in a fuzzy environment can be effectively used to optimize the operating modes of technological objects that are characterized by the fuzziness of some of the initial information. It should be noted that this requires the presence of experienced expert specialists, DMs, who control the operating modes of the object being optimized. The proposed approach was successfully tested when optimizing the operating modes of the technological complex for the production of the benzene Atyrau refinery and accepted for implementation.

6. Conclusions

This study developed simplex and heuristic methods for multi-criteria optimization of technological processes in a fuzzy environment based on modifications of search engine optimization methods and fuzzy set theory. Based on the results of this study, the following conclusions can be drawn:

(1) A method of multi-criteria optimization under fuzzy criteria based on the simplex method is proposed, based on the involvement of the DM to evaluate the vector of non-numerical criteria at the vertices of the simplex;

(2) A theorem has been formulated, and it has been proven that the resulting solution sequence based on the proposed simplex method is minimized. An estimate has been obtained that ensures that the convergence of the proposed simplex method increases with increasing size of the multi-criteria optimization problem being solved with fuzzy criteria;

(3) The formulation of the multi-criteria optimization problem in a fuzzy environment is formalized and obtained based on a modification and combination of the principles of maximin and Pareto optimality, and a heuristic method is developed for effectively solving the problem. The proposed heuristic method for solving a multi-criteria optimization problem in a fuzzy environment makes it possible to make maximum use of fuzzy information, i.e., knowledge, experience, and intuition of the DM, which ensures the adequacy of the solution and the effectiveness of the method;

(4) Using the proposed heuristic method, the problem of multi-criteria optimization of the benzene production process under Atyrau refinery conditions was solved. The advantage of the results of the proposed heuristic method compared to the results of known methods in solving the problem is shown. Based on the successful testing of

the developed heuristic method, Atyrau Refinery effectively applies it in the benzene production process.

The contribution of this study to the development of fuzzy optimization methods and the novelty of the proposed heuristic method for solving a multi-criteria optimization problem with fuzzy constraints is that, through the maximum use of fuzzy information, it is possible to obtain an effective solution to a fuzzy optimization problem. At the same time, the fuzzy information used when solving a multi-criteria optimization problem in a fuzzy environment represents the experience, knowledge, and intuition of the DM and experts involved in solving the problem.

The novelty of this study lies in the fact that the proposed heuristic method, due to the maximum use of collected fuzzy information, knowledge, experience, and intuition of the decision maker (DM) and experts, allows us to effectively solve a multi-criteria fuzzy problem in a fuzzy environment. In this case, the formalization of fuzzy information is carried out by constructing its membership function. The developed simplex optimization method in a fuzzy environment is based on the use of the decision maker's ability to fuzzily estimate the values of the criteria vector at the vertices of the simplex.

Possible limitations of the proposed heuristic method and ways to eliminate and mitigate them. Weaknesses such as subjectivity, cognitive constraints, and complexity of evaluation inherent in heuristic methods using the DM's preferences, knowledge, and intuition may limit its application in practice.

To eliminate and mitigate these weaknesses, we propose the following approaches, which we plan to implement in further research:

- Create and use a computer system that supports the evaluation process, working interactively with the DM. Such a computer system makes it possible to train and support the DM in the process of analysis and selection of the final solution. If necessary, it will be necessary to train and prepare the DM to work effectively on such a system;
- To solve the complexity of the lack of time for the DM to model and analyze various modes of operation of the object and compare the results in order to select the best solution can be eliminated by developing and using other heuristic methods. In this case, such principles of optimality and their combinations are necessary, which are more convenient and simple in the current production conditions;
- To reduce subjectivity in heuristic approaches, it is proposed to use a modified Delphi method operating in a computer network when conducting expert assessment. In this approach, by increasing the number of assessment rounds and gradually approaching the consistency of expert opinions in an iterative process, it is possible to ensure almost complete consistency of their opinions. In this case, the concordance coefficient will be approximately equal to 1, and subjectivity is significantly reduced. In addition, the network computer version significantly reduces the time required to collect, coordinate, and process expert information.

In our case, based on the analysis of the production situation, the presence of experienced DMs, experts in the field of the research object, and available information, the principles of maximin optimality and Pareto optimality were selected and modified for fuzziness as more suitable and convenient.

Author Contributions: Study conception and design: B.O., K.O. and Y.O.; data collection: S.S. and L.K.; analysis and interpretation of results: V.M., B.O. and Y.O.; draft manuscript preparation: Y.I. and A.O. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP23490206—Development of a decision support system for controlling the sulfur production process).

Data Availability Statement: Data is not available due to confidentiality restrictions. If specific data is required, it can be obtained from the author by correspondence to e-mail 78oea@mail.ru.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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