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The Approximation of Functions of Several Variables with Bounded p-Fluctuation by Polynomials in the Walsh System

Talgat Akhazhanov †, **Dauren Matin** *,† and **Zhuldyz Baituyakova**

Higher Mathematics Department, Faculty of Mechanics and Mathematics, L.N. Gumilyov Eurasian National University, Astana 010000, Kazakhstan; talgat-a2008@mail.ru (T.A.); baituyakova.zhzh@yandex.ru (Z.B.)
* Correspondence: d.matin@mail.kz; Tel.: +7-7773699966
† These authors contributed equally to this work.

Abstract: This paper presents direct and inverse theorems concerning the approximation of functions of several variables with bounded p-fluctuation using Walsh polynomials. These theorems provide estimates for the best approximation of such functions by polynomials in the norm of the space under consideration. The paper investigates the properties of the Walsh system, which includes piecewise constant functions, and builds on earlier work on trigonometric and multiplicative systems. The results are theoretical and have potential applications in such areas as coding theory, digital signal processing, pattern recognition, and probability theory.

Keywords: functions of bounded p-fluctuation; Walsh system; direct and converse theorems; discrete modulus of continuity

MSC: 42B20; 42B25

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1. Introduction

The Orthogonal Walsh system plays an important role in solving various theoretical and applied problems in mathematics. The classical theory of trigonometric series had a strong influence on the development of the theory of series in the Walsh system. In many issues, both similarities and significant differences were found. Unlike continuous trigonometric harmonics, the functions of the Walsh system comprise piecewise constant step functions. In this case, the Walsh functions take only two values, $+1$ and -1. Interest in the Walsh system arose due to the use of this system in applied issues (in coding theory, in digital signal processing, in pattern recognition, in probability theory, etc.).

It is known that the definition of the function of bounded p-fluctuation of one variable was introduced by Onneweer and Waterman [1]. In approximation theory, many scientific works are devoted to solving problems of the approximation of functions of one and several variables by polynomials in the Walsh system and partial Fourier–Walsh sums. Some results in this area can be found in articles [2–6].

The issues of obtaining direct and inverse theorems on some classes of functions were studied in the papers [7–11]. In a number of papers, estimates of the best approximation were obtained on some classes of functions of one variable defined using the modulus of continuity (see, for example, Refs. [9,11]). Volosivets in paper [3] began to study the approximation of functions of one variable with bounded fluctuations for multiplicative systems. Further, in papers [2–6], the authors of these studies obtained a number of exact estimates for a class of functions of one variable with bounded fluctuation for multiplicative systems.

In this paper, we continue to study these estimates for functions of several variables with bounded p-fluctuation for the Walsh system and related to approximative properties of polynomials with respect to the Walsh system in the class of functions of several variables with bounded p-fluctuation.

Let us give the necessary definitions.

Let $r_0(x)$ now be equal to 1 in $\left[0, \frac{1}{2}\right)$ and -1 in $\left[\frac{1}{2}, 1\right)$. We extend it periodically with the period 1 to the whole real line. The functions $r_0\left(2^k\right)$, $k = 1, 2, \ldots$ are called the Rademacher functions $r_k(x)$ [11]. Walsh functions in Paley enumeration are defined as follows. Set $w_0(x) \equiv 1$. For $n \in N$, consider the binary notation n:

$$n = \sum_{i=0}^{v} \epsilon_i 2^i,$$

where $\epsilon_v = 1$; $\epsilon_i = 0$ or $\epsilon_i = 1, 0 \leq i < v$. Then

$$w_n(x) = \prod_{i=0}^{v} (r_i(x))^{\epsilon_i}$$

is the n-th Walsh function [11].

Let $Z_+ = \{0, 1, 2, \ldots\}$. For $x \in [0, 1)$, the expansion holds

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$

where $x_k = 0$ or 1. This decomposition is determined uniquely if $x = \frac{k}{2^n}$, $0 < k < 2^n$, $k, n \in Z_+$. We take the decomposition with a finite number of $x_j \neq 0$.

For $x \in [0, 1)$, $y \in [0, 1)$,

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k}, y = \sum_{k=1}^{\infty} \frac{y_k}{2^k}$$

and the sum $x \oplus y$ is defined by the equality

$$x \oplus y = \sum_{k=1}^{\infty} \frac{(x_k + y_k)(mod 2)}{2^k}.$$

For fixed $x, y \in [0, 1)$, the equality

$$w_k(x \oplus y) = w_k(x) w_k(y), k \in Z_+$$

is true [11].

Let $d \in N$, $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in Z_+^d$, $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in [0, 1)^d$. Then, the multiple Walsh system is defined by the equality

$$w_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{d} w_{n_i}(x_i).$$

The Walsh system $\{w_{\mathbf{n}}(\mathbf{x})\}_{\mathbf{n} \in Z^d}$ is also orthonormal and complete in $L[0, 1)^d$ [11]. For $f \in L[0, 1)^d$, the Fourier coefficients are defined by

$$\widehat{f}(\mathbf{n}) = \int_0^1 \ldots \int_0^1 f(\mathbf{n}) w_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}, \mathbf{n} \in Z_+^d.$$

The space $L_p[0, 1)^d$, $1 \leq p \leq \infty$ is determined using the norm

$$\|f\|_{L_p[0,1)^d} = \begin{cases} \left(\int_0^1 \cdots \int_0^1 |f(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}, 1 \leq p < \infty \\ \sup\limits_{x \in [0,1)^d} |f(\mathbf{x})|, p = \infty \end{cases}$$

Let $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in Z_+^d$. Let us define sets of polynomials according to the Walsh system by

$$P_{\mathbf{n}} = \left\{ f \in L[0,1)^d : \widehat{f}(\mathbf{s}) = 0, s_{\mathbf{i}} \geq n_{\mathbf{i}}, i = 1, 2, \ldots, d \right\}$$

and

$$E_{\mathbf{n}}(f)_p = \inf\left\{ \|f - Q\|_p : Q \in P_{\mathbf{n}} \right\}$$

is the best approximation of functions by polynomials on the Walsh system. $E_n(f)_{p,\mathbf{m}}$ is defined in a similar way.

We denote by I_j^k dyadic interval $I_j^k = \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right)$, $j = 1, 2, \ldots, 2^k$ and let the function $f(\mathbf{x})$ be defined on the set $[0,1)^d$ and for any set I_j^k, denote by

$$I_{\mathbf{j}}^{(\mathbf{n})} = I_{j_1}^{n_1} \times \ldots \times I_{j_d}^{n_d}.$$

The Dirichlet kernel for the Walsh system is defined by the equality

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

The multiple Dirichlet kernel for the Walsh system is defined by the equality

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{d} D_{n_i}(x_i).$$

Lemma 1 ([11])**.** *For the Dirichlet kernel in the Walsh system, the equality holds*

$$D_{2^n} = 2^n \chi_{[0, \frac{1}{2^n})}, n \in Z_+$$

where $\chi_{[0, \frac{1}{2^n})}$ *is the characteristic function on the interval* $\left[0, \frac{1}{2^n}\right)$.

Lemma 2. *Let* $f(\boldsymbol{x}) \in L_1(I_{\boldsymbol{j}}^{(\boldsymbol{n})})$*. Then, the equality holds*

$$S_{2^{j_1}, \ldots, 2^{j_d}} f(\boldsymbol{x}) = 2^{j_1 + \ldots + J_d} \int_{I_{\boldsymbol{j}}^{(\boldsymbol{n})}} f(\boldsymbol{t}) d\boldsymbol{t}$$

for some $\boldsymbol{x} \in I_{\boldsymbol{j}}^{(\boldsymbol{n})}$*, where* $\boldsymbol{n} = (n_1, \ldots, n_d)$, $d\boldsymbol{t} = (dt_1, \ldots, dt_d)$, $\int_{I_{\boldsymbol{j}}^{(\boldsymbol{n})}} = \int_{I_{n_1}^{J_1}} \ldots \int_{I_{n_d}^{j_d}}$.

Proof. In the case of functions of one variable, such a statement is available (see [11], p. 27). In the expression for the partial sum of the series, we substitute the value of the Fourier–Walsh coefficients:

$$S_{\mathbf{n}}(\mathbf{x}) = \sum_{i_1=1}^{n_1-1} \ldots \sum_{i_d=1}^{n_d-1} \int_{I_{\mathbf{j}}^{(\mathbf{n})}} w_{\mathbf{i}}(\mathbf{t}) f(\mathbf{t}) w_{\mathbf{i}}(\mathbf{x}) d\mathbf{t} = \int_{I_{\mathbf{j}}^{(\mathbf{n})}} w(\mathbf{t}) \left(\sum_{i_1=1}^{n_1-1} \ldots \sum_{i_d=1}^{n_d-1} w_{\mathbf{i}}(\mathbf{t} \oplus \mathbf{x}) \right) d\mathbf{t}.$$

From the definition of the Dirichlet kernel and the invariance of the integral with respect to a shift, we obtain

$$S_{\mathbf{n}}(\mathbf{x}) = \int_{I_{\mathbf{j}}^{(\mathbf{n})}} f(\mathbf{t}) D_{\mathbf{n}}(\mathbf{t} \oplus \mathbf{x}) d\mathbf{t} = \int_{I_{\mathbf{j}}^{(\mathbf{n})}} f(\mathbf{t} \oplus \mathbf{x}) D_{\mathbf{n}}(\mathbf{t}) d\mathbf{t}.$$

In particular, for partial sums with $n_i = 2^j, i = 1, 2, \ldots, d$, we can write

$$S_{2^j,\dots,2^j}f(\mathbf{x}) = \int_{I_{\mathbf{j}}^{(\mathbf{n})}} D_{2^j,\dots,2^j}(\mathbf{t})f(\mathbf{t}\oplus\mathbf{x})d\mathbf{t} = 2^{dj}\int_{I_{\mathbf{j}}^{(\mathbf{n})}} f(\mathbf{t}\oplus\mathbf{x})d\mathbf{t}.$$

Further, if $\mathbf{x},\mathbf{t} \in I_{\mathbf{j}}^{(\mathbf{n})}$, then the following is true:

$$\mathbf{t}\oplus\mathbf{x} \in I_{\mathbf{j}}^{(\mathbf{n})}$$

following

$$S_{2^j,\dots,2^j}f(\mathbf{x}) = 2^{dj}\int_{I_{\mathbf{j}}^{(\mathbf{n})}} f(\mathbf{t})d\mathbf{t},$$

for some $\mathbf{x} \in I_{\mathbf{j}}^{(\mathbf{n})}$, where $\mathbf{t} = (t_1,\dots,t_d)$, $d\mathbf{t} = (dt_1,\dots,dt_d)$.

Lemma 2 has been proven. □

The oscillation of the function $f(\mathbf{x})$ on interval $I_{\mathbf{j}}^{(\mathbf{n})}$ is defined by the equality

$$osc(f, I_{\mathbf{i}}^{\mathbf{n}}) = \sup_{(\mathbf{x},\mathbf{y})\in I_{\mathbf{i}}^{\mathbf{n}}} |f(\mathbf{x}) - f(\mathbf{y})|$$

Let $1 \le p < \infty$ and the function $f(\mathbf{x})$ be defined on $I_{\mathbf{j}}^{(\mathbf{n})}$; we define an oscillatory sum of order $(2^{n_1},\dots,2^{n_d})$ by the equality

$$\kappa_p(f,\mathbf{n}) := \left(\sum_{j_1=1}^{2^{n_1}}\dots\sum_{j_d=1}^{2^{n_d}}\left(osc\left(f, I_{\mathbf{j}}^{(\mathbf{n})}\right)\right)^p\right)^{1/p}.$$

If

$$V_p^{\mathbf{d}}(f) := \sup_{n_{\mathbf{k}}\in Z_+} \kappa_p(f, n_1,\dots,n_d) < \infty,$$

then $f(\mathbf{x})$ is called the function of bounded p-fluctuation.

Now, we introduce a fluctuation modulus of continuity

$$V_p^d(f)_{\mathbf{m}} = \sup_{\mathbf{n}_j\ge\mathbf{m}_j} \kappa_p(f, n_1,\dots,n_d), j = 1,2,\dots,d.$$

In one variable case, the definition was introduced in [3]. We denote by $MV_p[0,1)^d$ the set of functions $f(\mathbf{x})$, for which $V_p^d(f) < \infty$, $(1 \le p < \infty)$, and by $MC_p[0,1)^d$, $(1 < p < \infty)$ the set of functions $f(\mathbf{x})$, for which $V_p^d(f) < \infty$,

$$V_p^d(f)_{\mathbf{n}} \to 0$$

when $n_k \to \infty$, $k = 1,\dots,d$. In addition, these spaces are considered for indexed boundednesses with the norm

$$\|f\|_{p,\mathbf{m}} = max\left(V_p^d(f), \|f\|_\infty\right).$$

Let us introduce one more discrete group modulus of continuity related to the space $MC_p[0,1)^d$ $(1 < p < \infty)$, by the formula

$$V_p^d(f)^*_{\mathbf{n}_k} = \sup_{0\le h_k<\frac{1}{2^{n_k}}} \|f(x_1\oplus h_1,\dots,x_k\oplus h_d) - f(x_1,\dots,x_k)\|_{p,\mathbf{m}}, k = 1,2,\dots,d.$$

In one variable case, the definition was introduced in Ref. [3].

Lemma 3. *The spaces $MV_p[0,1)^d$ and $MC_p[0,1)^d$ are complete relative to the norm*

$$\|f\|_{p,\boldsymbol{m}} = max\left(V_p^d(f), \|f\|_\infty\right)$$

Proof. Without loss of generality, we consider only the case of functions of two variables (d = 2). Let $\{f_{k,l}\}_{k,l=1}^{\infty} \subset MV_p[0,1)^2$ be a fundamental sequence by norm $\|f\|_{p,\mathbf{m}}$. Then, for any $\varepsilon > 0$, there are $N_\varepsilon, K_\varepsilon \in N$ such that $\left\|f_{l,k} - f_{i,j}\right\|_\infty \le \varepsilon$ and $\sup\limits_{n_1,n_2 \in P} \kappa_p(f_{l,k} - f_{i,j}, n_1, n_2) \le \varepsilon$ for all $k, i \ge N_\varepsilon$, $l, j \ge K_\varepsilon$.

Due to the completeness of the space $B[0,1)^2$, the sequence of bounded functions $\{f_{i,j}\}$ on $B[0,1)^2$ with the norm $\|f\|_\infty$ converges uniformly to some $f \in B[0,1)^2$. Taking into account the finiteness of the number of terms in $\kappa_p(f_{l,k} - f_{i,j}, n_1, n_2)$ and taking the limit as $k, l \to \infty$, we obtain $\left\|f_{l,k} - f_{i,j}\right\|_\infty \le \varepsilon$ and $\kappa_p(f_{l,k} - f_{i,j}, n_1, n_2) \le \varepsilon$ for all $k, i \ge N_\varepsilon, l, j \ge K_\varepsilon$.

This means that $\left\|f_{l,k} - f_{i,j}\right\|_\infty \le \varepsilon$ for all $k, i \ge N_\varepsilon, l, j \ge K_\varepsilon$. Since

$$V_p(f) \ge V_p(f - f_{i,j}) + V_p(f_{i,j}) \le \varepsilon + V_p(f_{i,j})$$

then the limit function $f(x,y)$ has a bounded p-fluctuation and the space $MV_p[0,1)^2$ is complete.

To prove completeness of $MC_p[0,1)^2$, we must show that if $V_p(f_{m,n})_{n_1,n_2} \to 0$ for $n_1 \to \infty, n_2 \to \infty$ and $\{f_{m,n}\}_{m,n=1}^{\infty}$ converges to f in $MC_p[0,1)^2$, then $f \in MC_p[0,1)^2$. It is obvious that

$$V_p(f, n_1, n_2) \ge V_p(f - f_{m,n}, n_1, n_2) + V_p(f, n_1, n_2)$$

By choosing m and n such that

$$V_p(f - f_{m,n}, n_1, n_2) < \frac{\varepsilon}{2},$$

and selecting m_0 and n_0 such that

$$V_p(f_{m,n}, n_1, n_2) < \frac{\varepsilon}{2}$$

for $m > m_0, n > n_0$, we conclude that

$$V_p(f, n_1, n_2) < \varepsilon$$

for $m > m_0, n > n_0$.

Lemma 3 has been proven. □

2. Direct and Converse Theorems

The goal of this work is to obtain direct and converse theorems for functions from the space $MV_p[0,1)^d$ by means of the Walsh polynomials in the $\|\|_{p,\mathbf{m}}$ norm.

Firstly, we will prove Lemma 4, which is necessary to prove Theorem 1.

Lemma 4. *Let* $1 \le p < \infty$, $\varphi(\boldsymbol{x}, \boldsymbol{y}) \in MV_p[0,1)^d$ *for* $(\boldsymbol{y}) \in [0,1)^d$. *Then, the following inequality is valid:*

$$\left\| \int_0^1 \dots \int_0^1 \varphi(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y} \right\|_{p,\boldsymbol{m}} \le \int_0^1 \dots \int_0^1 \|\varphi(\boldsymbol{x}, \boldsymbol{y})\|_{p,\boldsymbol{m}} d\boldsymbol{y}.$$

Proof. Without loss of generality, we consider only the case of functions of two variables (d = 2). Let

$$h(x_1, x_2) = \int_0^1 \int_0^1 \varphi(x_1, x_2, y_1, y_2) dy_1 dy_2.$$

Consider the sum $\kappa_p(h, n_1, n_2)$ as L_p—the norm of the function $Q(x_1, x_2, t_1, t_2)$ equal to

$$osc\left(h, I_{j_1,j_2}^{(n_1,n_2)}\right) \times 2^{(n_1+n_2)\frac{1}{p}}$$

on $I_{j_1,j_2}^{(n_1,n_2)} \times [0,1)^2$. By definition,

$$\kappa_p(h,n_1,n_2) = \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left(\sup_{\substack{(x_1,x_2)\in I_{j_1,j_2}^{(n_1,n_2)} \\ (x_1',x_2')\in I_{j_1,j_2}^{(n_1,n_2)}}} \left| h(x_1,x_2) - h(x_1',x_2') \right| \right)^p \right)^{\frac{1}{p}}.$$

Then,

$$\|Q\|_{L_p} = \left(\int_0^1 \int_0^1 |Q(x_1,x_2,t_1,t_2)|^p dx_1 dx_2 \right)^{\frac{1}{p}}$$

$$= \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \int_{I_{j_1}^{(n_1)}} \int_{I_{j_2}^{(n_2)}} \left| osc\left(h, I_{j_1,j_2}^{(n_1,n_2)}\right) 2^{(n_1+n_2)\frac{1}{p}} \right|^p dx_1 dx_2 \right)^{\frac{1}{p}}$$

$$= \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left| osc\left(h, I_{j_1,j_2}^{(n_1,n_2)}\right) \right|^p \right)^{\frac{1}{p}} 2^{(n_1+n_2)} \int_{I_{j_1}^{(n_1)}} \int_{I_{j_2}^{(n_2)}} dx_1 dx_2$$

$$= \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left| osc\left(h, I_{j_1,j_2}^{(n_1,n_2)}\right) \right|^p \right)^{\frac{1}{p}} = \kappa_p(h,n_1,n_2).$$

then,

$$\|Q\|_{L_p} = \kappa_p(h,n_1,n_2).$$

According to the definition,

$$\left\| \int_0^1 \int_0^1 \varphi(x_1,x_2,y_1,y_2) dy_1 dy_2 \right\|_{p,\mathbf{m}} = max\left(V_p(h), \|h\|_\infty\right).$$

Next, applying the generalized Minkowsk inequality for L_p, we have

$$V_p(h) = \sup_{\overline{n}\in P} \kappa_p(h,\overline{n}) = \sup_{\overline{n}\in P} \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \sup_{\overline{x},\overline{x'}\in I_{\overline{j}}^{(\overline{n})}} \left| \int_0^1 \int_0^1 \left(\varphi(\overline{x},\overline{y}) - \varphi\left(\overline{x'},\overline{y'}\right) \right) d\overline{y} \right|^p \right)^{\frac{1}{p}}$$

$$= \sup_{\overline{n}\in P} \left(\sup_{\overline{x},\overline{x'}\in I_{\overline{j}}^{(\overline{n})}} \sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left| \int_0^1 \int_0^1 \left(\varphi(\overline{x},\overline{y}) - \varphi\left(\overline{x'},\overline{y'}\right) \right) d\overline{y} \right|^p \right)^{\frac{1}{p}}$$

$$\le \sup_{\overline{n}\in P} \sup_{\overline{x},\overline{x'}\in I_{\overline{j}}^{(\overline{n})}} \int_0^1 \int_0^1 \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left| \left(\varphi(\overline{x},\overline{y}) - \varphi\left(\overline{x'},\overline{y'}\right) \right) \right|^p \right)^{\frac{1}{p}} d\overline{y}$$

$$\le \int_0^1 \int_0^1 \left(\sup_{\overline{n}\in P} \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \sup_{\overline{x},\overline{x'}\in I_{\overline{j}}^{(\overline{n})}} \left| \left(\varphi(\overline{x},\overline{y}) - \varphi\left(\overline{x'},\overline{y'}\right) \right) \right|^p \right)^{\frac{1}{p}} \right) d\overline{y}$$

$$= \int_0^1 \int_0^1 \|\varphi(x_1,x_2,y_1,y_2)\|_{p,\mathbf{m}} dy_1 dy_2.$$

In addition,

$$\sup_{\overline{x},\overline{y}\in[0,1]^2}\left(\int_0^1\int_0^1 \varphi(\overline{x},\overline{y})d\overline{y}\right) \le \int_0^1\int_0^1\left(\sup_{\overline{x},\overline{y}\in[0,1]^2}\varphi(\overline{x},\overline{y})\right)d\overline{y}.$$

$A_1 < B_1, A_2 < B_2 \Rightarrow max(A_1, A_2) \le max(B_1, B_2)$. Lemma 4 is proven. □

Theorem 1. *Let* $1 < p < \infty$, $n_i \in N$, $f \in MV_p[0,1)^d$. *Then, the following inequality is valid:*

$$\frac{1}{2}V_p^d(f)^*_{\boldsymbol{n}} \le E_{2^{n_1},\dots,2^{n_d}}(f)_{p,\boldsymbol{m}} \le V_p^d(f)^*_{\boldsymbol{n}}.$$

Proof. Without loss of generality, we consider only the case of functions of two variables (d = 2). Applying Lemma 2, we obtain

$$S_{2^m,2^n}(f,x,y) = 2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} f(x\oplus u, y\oplus v)dudv.$$

$$E_{2^m,2^n}(f)_{p,F} \le \|f(x,y) - S_{2^m,2^n}(f,x,y)\|_{p,\mathbf{m}}$$

$$= \left\|f(x,y) - 2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} f(x\oplus u, y\oplus v)dudv\right\|_{p,\mathbf{m}}$$

$$= \left\|2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} (f(x,y) - f(x\oplus u, y\oplus v))dudv\right\|_{p,\mathbf{m}}$$

$$= max\left(V_p^2\left(2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} (f(x,y) - f(x\oplus u, y\oplus v))dudv\right),\right.$$

$$\left.\left\|2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} (f(x,y) - f(x\oplus u, y\oplus v))dudv\right\|_{\infty}\right)$$

We denote

$$g(x,y) = 2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}} (f(x,y) - f(x\oplus u, y\oplus v))dudv.$$

When applying the definition of the $V_p(g)$ and Lemma 4, we have

$$V_p^2(g)$$

$$= 2^{m+n}\sup_{(k_1,k_2)\in P^2}\left(\sum_{j_1=1}^{2^{k_1}}\sum_{j_2=1}^{2^{k_2}}\sup_{\substack{(x,y)\in I_{j_1,j_2}^{(k_1,k_2)}\\(x',y')\in I_{j_1,j_2}^{(k_1,k_2)}}}\left(\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}}|f(x,y) - f(x\oplus u, y\oplus v)-\right.\right.$$

$$\left.\left.-\big(f(x',y') - f(x'\oplus u, y'\oplus v)\big)\big|dudv\right)^p\right)^{\frac{1}{p}}$$

$$\le 2^{m+n}\int_0^{\frac{1}{2^m}}\int_0^{\frac{1}{2^n}}\left(\sup_{(k_1,k_2)\in P^2}\sum_{j_1=1}^{2^{k_1}}\sum_{j_2=1}^{2^{k_2}}\left(\sup_{\substack{(x,y)\in I_{j_1,j_2}^{(k_1,k_2)}\\(x',y')\in I_{j_1,j_2}^{(k_1,k_2)}}}|f(x,y) - f(x\oplus u, y\oplus v)-\right.\right.$$

$$-\big(f(x',y') - f(x'\oplus u, y'\oplus v)\big)\big|\big)^p\Big)^{\frac{1}{p}} dudv$$

$$\leq 2^{m+n} \sup_{\substack{u\in\left[0,\frac{1}{2^m}\right)\\ v\in\left[0,\frac{1}{2^n}\right)}} \sup_{(k_1,k_2)\in P^2} \left(\sum_{j_1=1}^{2^{k_1}} \sum_{j_2=1}^{2^{k_2}} \left(\sup_{\substack{(x,y)\in I_{j_1,j_2}^{(k_1,k_2)}\\ (x',y')\in I_{j_1,j_2}^{(k_1,k_2)}}} |f(x,y) - f(x\oplus u, y\oplus v) - \right.\right.$$

$$\left.\left. -\big(f(x',y') - f(x'\oplus u, y'\oplus v)\big)\big|\big)^p\right)^{\frac{1}{p}} \int_0^{\frac{1}{2^m}} \int_0^{\frac{1}{2^n}} dudv = V_p^2(f)^*_{m,n}.$$

In addition, it holds

$$E_{2^m,2^n}(f)_\infty = \inf_{\substack{k_1\leq 2^m\\ k_2\leq 2^n}} \sup_{(x,y)\in[0,1)^2} |f(x,y) - P_k(x,y)| \leq \sup_{(x,y)\in[0,1)^2} |f(x,y) - S_{2^m,2^n}(x,y)|$$

$$= \sup_{(x,y)\in[0,1)^2} \left| f(x,y) - 2^{m+n} \int_0^{\frac{1}{2^m}} \int_0^{\frac{1}{2^n}} f(x\oplus u, y\oplus v) dudv \right|$$

$$= \sup_{(x,y)\in[0,1)^2} \left| 2^{m+n} \int_0^{\frac{1}{2^m}} \int_0^{\frac{1}{2^n}} (f(x,y) - f(x\oplus u, y\oplus v)) dudv \right|$$

$$\leq \sup_{(x,y)\in[0,1)^2} \sup_{(u,v)\in\left[0,\frac{1}{2^m}\right)\times\left[0,\frac{1}{2^n}\right)} |(f(x,y) - f(x\oplus u, y\oplus v))|$$

$$\leq \sup_{(u,v)\in\left[0,\frac{1}{2^m}\right)\times\left[0,\frac{1}{2^n}\right)} \sup_{(x,y)\in[0,1)^2} |(f(x,y) - f(u, y\oplus v))| \leq V(f)^*_{m,n}.$$

Then,

$$E_{2^m,2^n}(f)_{p,\mathbf{m}} \leq V(f)^*_{m,n}.$$

Now, let us prove the left inequality from the theorem. For this, we first note that from the properties of the Walsh system it follows that

$$0 \leq h_1 \leq \frac{1}{2^m}, 0 \leq h_2 \leq \frac{1}{2^n}$$

and with all

$$k_1 = 0, 1, \ldots, 2^m - 1, k_2 = 0, 1, \ldots, 2^n - 1.$$

we have

$$w_{k_1,k_2}(x\oplus h_1, y\oplus h_2) = w_{k_1}(x) w_{k_2}(y).$$

Let

$$Q_{2^m,2^n}(f,x,y) = \sum_{k_1=0}^{2^m-1} \sum_{k_2=0}^{2^n-1} a_{k_1,k_2} w_{k_1}(x) w_{k_2}(y)$$

be the polynomial that best approximates the function f in the metric $p, \mathbf{m}$,

$$E_{2^m,2^n}(f)_{p,\mathbf{m}} = \|f - Q_{2^m,2^n}\|_{p,\mathbf{m}}.$$

From the above, it follows that

$$Q_{2^m,2^n}(f,x,y) = Q_{2^m,2^n}(f, x\oplus h_1, y\oplus h_2).$$

Therefore, for any $0 \leq h_1 \leq \frac{1}{2^m}, 0 \leq h_2 \leq \frac{1}{2^n}$, we have

$$\|f(x,y) - f(x\oplus h_1, y\oplus h_2)\|_{p,\mathbf{m}}$$

$$= \|f(x,y) - Q_{2^m,2^n}(f,x,y) + Q_{2^m,2^n}(f,x\oplus h_1,y\oplus h_2) - f(x\oplus h_1,y\oplus h_2)\|_{p,\mathbf{m}}$$
$$\leq \|f(x,y) - Q_{2^m,2^n}(f,x,y)\|_{p,\mathbf{m}} + \|Q_{2^m,2^n}(f,x\oplus h_1,y\oplus h_2) - f(x\oplus h_1,y\oplus h_2)\|_{p,\mathbf{m}}$$
$$= E_{2^m,2^n}(f)_{p,\mathbf{m}} + E_{2^m,2^n}(f)_{p,\mathbf{m}} = 2E_{2^m,2^n}(f)_{p,\mathbf{m}}.$$

Therefore,

$$\sup_{\substack{0\leq h_1\leq \frac{1}{2^m}\\ 0\leq h_2\leq \frac{1}{2^n}}} \|f(x,y) - f(x\oplus h_1,y\oplus h_2)\|_{p,\mathbf{m}} \leq 2E_{2^m,2^n}(f)_{p,\mathbf{m}}.$$

So,

$$V(f)^*_{m,n} \leq 2E_{2^m,2^n}(f)_{p,\mathbf{m}}.$$

The theorem is proven. □

Theorem 2. *Let $1 < p < \infty$, $n_i \in N$, $f \in MV_p[0,1)^d$. Then, the following inequality is valid:*

$$V_p(f)_{\boldsymbol{n}} \leq E_{2^{n_1},\ldots,2^{n_d}}(f)_{p,\boldsymbol{m}}.$$

Proof. Without loss of generality, we consider only the case of functions of two variables (d = 2). To prove the left inequality, we note that any polynomial $Q_{2^m,2^n} \in W_{2^m,2^n}$ is constant on any rectangle $I^{(k_1)}_{j_1} \times I^{(k_2)}_{j_2}$, $j_1 = 1,2,\ldots,2^{k_1}$, $j_2 = 1,2,\ldots,2^{k_2}$, $k_1 \geq m$, $k_2 \geq n$; so,

$$osc\Big(f, I^{(k_1,k_2)}_{j_1,j_2}\Big) = osc\Big(f - Q_{2^m,2^n}, I^{(k_1,k_2)}_{j_1,j_2}\Big)$$

and for $k_1 \geq m$, $k_2 \geq n$, corresponding sums $\kappa_p(f - Q_{2^m,2^n}, k_1, k_2)$ and $\kappa_p(f, k_1, k_2)$ match. Hence,

$$E_{2^m,2^n}(f)_{p,\mathbf{m}} \geq \sup_Q \|f - Q_{2^m,2^n}\|_{p,\mathbf{m}}$$
$$= max\Big(V_p(f - Q_{2^m,2^n}), \|f - Q_{2^m,2^n}\|_{p,\mathbf{m}}\Big) \geq V_p(f - Q_{2^m,2^n}),$$

then,

$$E_{2^m,2^n}(f)_{p,\mathbf{m}} \geq V_p(f)_{m,n}.$$

The theorem is proven. □

Note that the direct and converse theorems of the theory of the approximation of functions of bounded p-variation by polynomials with respect to multiplicative systems are considered in paper [3].

Remark 1. *When proving theorems, we used methods of metric theory of functions and methods of approximation theory.*

3. Conclusions

In this paper, we proved direct and converse theorems of the approximation of functions of several variables of bounded p-fluctuations by Walsh polynomials in the norm of the considered space. Estimates of the best approximation through the fluctuation modulus of continuity and an estimate of the fluctuation modulus of continuity through the best approximation have been obtained for the classes of functions of several variables, $MV_p[0,1)^d$. The results of this work are theoretical and can be applied in the theory of orthogonal series, approximation theory, and harmonic analyses. They can be used in coding theory, in digital signal processing, in pattern recognition, in probability theory, etc.

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