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ALTERNATIVE CRITERIA FOR BOUNDEDNESS OF ONE CLASS OF MATRIX OPERATORS IN WEIGHTED SPACES OF SEQUENCES

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Abstract. Characterizations of weighted integral Hardy inequalities identify the weights ensuring the boundedness of the Hardy operator on weighted Lebesgue spaces. This analysis has been extended to the Riemann-Liouville operator and operators satisfying a weaker kernel condition, independently introduced by R. Oinarov [13] and by S. Bloom and R. Kerman [5]. While Oinarov-type characterizations have been extended to the discrete case, Bloom-Kerman-type characterizations, which differ, remain unexplored. This paper establishes these alternative characterizations for the discrete case and extends the results to a broader class of matrix operators, including those satisfying weaker kernel conditions.

1. Introduction

For $f \geqslant 0$ the weighted integral inequality

$$\left(\int_0^\infty u^q(x)\left(\int_0^x f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty v^p(x) f^p(x)\,dx\right)^{\frac{1}{p}} \tag{1}$$

is a generalization of the famous Hardy inequality. The validity of the inequality (1) is equivalent to characterization of weight functions $u(x)$ and $v(x)$, for which the Hardy operator $Hf(x) = \int_0^x f(s)\,ds$ is bounded on weighted Lebesgue spaces. This problem was completely solved for all possible relations between the parameters p and q (for more details we refer to [11, Theorem 5] and [12]). The replacement of the Hardy operator by the Riemann-Liouville operator was a further important generalization of the inequality (1) introduced by V. D. Stepanov in the works [20, 21, 22].

The boundedness on weighted Lebesgue spaces of a more general operator $Kf(x) = \int_0^x K(x,s)f(s)ds$ with a kernel $K(\cdot,\cdot)$ satisfying the condition: $K(x,s) \geqslant 0$ for $x \geqslant s > 0$ and there exists a number $d > 1$ such that

$$\frac{1}{d}\big(K(x,t)+K(t,s)\big) \leqslant K(x,s) \leqslant d\big(K(x,t)+K(t,s)\big), \ \ \forall\, x \geqslant t \geqslant s > 0, \tag{2}$$

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was independently established in [13] by R. Oinarov and in [5] by S. Bloom and R. Kerman. Thus, for $1 < p \leqslant q < \infty$ and $K(\cdot,\cdot)$ satisfying the condition (2), in [13] R. Oinarov found that the inequality

$$\left(\int_0^\infty u^q(x)\left(\int_0^x K(x,s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty v^p(x)f^p(x)\,dx\right)^{\frac{1}{p}} \tag{3}$$

holds if and only if

$$\sup_{z>0}\left(\int_z^\infty K^q(t,z)u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty, \tag{4}$$

$$\sup_{z>0}\left(\int_z^\infty u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z K^{p'}(z,s)v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty, \tag{5}$$

while in [5] S. Bloom and R. Kerman proved that it holds if and only if

$$\sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K^q(t,s)u^q(t)dt\right)^{p'} ds\right)^{\frac{1}{p'}}\left(\int_z^\infty K^q(t,z)u^q(t)dt\right)^{-\frac{1}{q'}} < \infty, \tag{6}$$

$$\sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(t,s)u^q(t)dt\right)^{p'} ds\right)^{\frac{1}{p'}}\left(\int_z^\infty u^q(t)dt\right)^{-\frac{1}{q'}} < \infty. \tag{7}$$

There are numerous works using operators with kernels satisfying (2) in various problems of Analysis, the most of which refer to both of the papers [13] and [5]. However, currently the condition (2) is called the Oinarov condition.

Let us turn to the discrete case. For nonnegative sequences $f = \{f_i\}_{i=1}^\infty$ the weighted discrete inequality has the form

$$\left(\sum_{i=1}^\infty u_i^q\left(\sum_{j=1}^i a_{i,j}f_j\right)^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^\infty v_i^p f_i{}^p\right)^{\frac{1}{p}}, \tag{8}$$

where $u = \{u_i\}_{i=1}^\infty$ and $v = \{v_i\}_{i=1}^\infty$ are sequences of positive real numbers and

$$(Af)_i = \sum_{j=1}^i a_{i,j}f_j \tag{9}$$

is a matrix operator with a lower triangular matrix (a_{ij}), $i \geqslant j$, $i,j \in \mathbb{N}$.

Similarly to the integral case, for $a_{i,j} \equiv 1$ the operator (9) is the discrete Hardy operator, and the weighted estimate (8) for this operator was studied for all relations between the parameters p and q (see [1, 2, 3, 4, 6, 7]).

In the paper [17], there was introduced a discrete analogue of the Oinarov condition: $a_{i,j} \geqslant 0$ for $i \geqslant j$ and there exists a number $d > 1$ such that

$$\frac{1}{d}(a_{i,k}+a_{k,j}) \leqslant a_{i,j} \leqslant d(a_{i,k}+a_{k,j}), \ \ \forall\, i \geqslant k \geqslant j \geqslant 1. \tag{10}$$

Then it was proved that for $1 < p \leqslant q < \infty$ and $a_{i,j}$ satisfying the condition (10) the inequality (8) holds if and only if

$$\sup_{i\geqslant 1}\left(\sum_{n=i}^{\infty} a_{n,i}^{q} u_n^{q}\right)^{\frac{1}{q}}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{1}{p'}} < \infty,$$

$$\sup_{i\geqslant 1}\left(\sum_{n=i}^{\infty} u_n^{q}\right)^{\frac{1}{q}}\left(\sum_{j=1}^{i} a_{i,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

It is obvious that the above two conditions are analogues of the characterizations (4) and (5) for the integral inequality (3). Wondering whether we could obtain discrete analogues of the conditions (6) and (7), we have found an alternative criterion for the validity of the inequality (8) and proved the following theorem.

THEOREM 1. *Let $1 < p \leqslant q < \infty$ and a matrix $(a_{i,j})$ satisfy the condition (10). Then the inequality (8) holds if and only if $A = \max\{A_1, A_2\} < \infty$, where*

$$A_1 = \sup_{k\geqslant 1}\left(\sum_{n=k}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n}^{q} u_i^{q}\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty} a_{n,k}^{q} u_n^{q}\right)^{-\frac{1}{q'}},$$

$$A_2 = \sup_{k\geqslant 1}\left(\sum_{n=k}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n} u_i^{q}\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=k}^{\infty} u_n^{q}\right)^{-\frac{1}{q'}}.$$

Moreover, $C \approx A$, where C is the best constant in (8).

In order not to simply transfer the results of S. Bloom and R. Kerman from [5] to the discrete case, we find necessary and sufficient conditions for the validity of the inequality (8) for broader classes of matrix operators introduced in [18]. This ensures that Theorem 1 becomes a corollary of the main result (see Theorem 2). These classes are also discussed in the recent papers [10] and [23]. The integral version of these classes was introduced in [14], with the most recent publications using them being [8] and [19]. Since these classes of integral and discrete operators are broader than the classes of operators satisfying the Oinarov conditions (2) and (10), we believe they will be the subject of many future publications.

Note that the inequality (8) can be rewritten shorter in the form

$$\|Af\|_{q,u} \leqslant C\|f\|_{p,v}, \tag{11}$$

where $\|f\|_{p,v} \equiv \|vf\|_p = \left(\sum\limits_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}$, $1 \leqslant p < \infty$, is the norm of the weighted space $l_{p,v}$ of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers. Moreover, the validity of the inequality (11) is equivalent to the boundedness of the matrix operator (9) from $l_{p,v}$ into $l_{q,u}$.

For a more complete presentation, we also consider the dual inequality

$$\|A^* f\|_{q,u} \leqslant C\|f\|_{p,v}, \tag{12}$$

where

$$(A^* f)_n = \sum_{i=n}^{\infty} a_{i,n} f_i, \;\; \forall\, n \in \mathbb{N}. \tag{13}$$

The paper is organized as follows: In Section 2 we provide definitions of classes of matrix operators, denoted $\mathscr{O}_1^+$ and $\mathscr{O}_1^-$. Moreover, there we also collect some auxiliary statements that we need to prove the main results. Section 3 presents criteria for the validity of the inequality (11) when the matrix operator (9) belongs to the class $\mathscr{O}_1^+$. In Section 4 we obtain similar results for the operator (9) when it belongs to the class $\mathscr{O}_1^-$.

2. Auxiliary definitions and statements

In [18] R. Oinarov and Z. Taspaganbetova introduced extended classes of matrices $\mathscr{O}_n^{\pm}$, $n > 0$, and obtained criteria for the fulfillment of the inequality (11) for matrix operators from these classes. The class $\mathscr{O}_1^+$ includes matrices $(a_{i,j})$ satisfying the following condition: $a_{i,j} \geqslant 0$ for $i \geqslant j$ and there exist a number $d > 1$, a sequence of positive numbers $\{w_k\}_{k=1}^{\infty}$ and a nonnegative matrix $(b_{i,j})$, where $b_{i,j}$ does not decrease in the first index i and does not increase in the second index j, such that

$$\frac{1}{d}(b_{i,k}w_j + a_{k,j}) \leqslant a_{i,j} \leqslant d(b_{i,k}w_j + a_{k,j}), \;\; \forall\, i \geqslant k \geqslant j \geqslant 1. \tag{14}$$

The class $\mathscr{O}_1^-$ includes matrices $(a_{i,j})$ satisfying the following condition: $a_{i,j} \geqslant 0$ for $i \geqslant j$ and there exist a number $d > 1$, a sequence of positive numbers $\{w_k\}_{k=1}^{\infty}$ and a nonnegative matrix $(b_{i,j})$, where $b_{i,j}$ does not decrease in the first index i and does not increase in the second index j, such that

$$\frac{1}{d}(a_{i,k} + b_{k,j}w_i) \leqslant a_{i,j} \leqslant d(a_{i,k} + b_{k,j}w_i), \;\; \forall\, i \geqslant k \geqslant j \geqslant 1. \tag{15}$$

Note that the conditions (14) and (15) cover the discrete analogue of the Oinarov condition (10) and complement each other. It is not difficult to show that the classes $\mathscr{O}_1^+$ and $\mathscr{O}_1^-$ are wider than the class of matrices satisfying the Oinarov discrete condition, since for $w_k \equiv 1$ and $b_{i,j} \equiv a_{i,j}$ from (14) and (15) it follows (10), i.e., matrices satisfying the condition (10) belong to the class $\mathscr{O}_1^+ \bigcap \mathscr{O}_1^-$.

The following simple example illustrates that e.g. the condition (14) is weaker than the discrete Oinarov condition (10). Suppose that for all $i \geqslant k \geqslant 1$ matrices $\widetilde{a}_{i,k}$ satisfy the discrete Oinarov condition (10), then $a_{i,j} = \sum_{k=1}^{j} \widetilde{a}_{i,k}$ belong to the class $\mathcal{O}_1^+$. Indeed, for $i \geqslant n \geqslant j \geqslant 1$ we have that $a_{i,j} \approx \sum_{k=1}^{j} (\widetilde{a}_{i,n} + \widetilde{a}_{n,k}) = \widetilde{a}_{i,n} \cdot j + \sum_{k=1}^{j} \widetilde{a}_{n,k} = \widetilde{a}_{i,n} \cdot j + a_{n,j}$. However, matrices $a_{i,j}$ do not satisfy the Oinarov condition.

The following theorems were proved in [15] and [24], respectively (see also [18] for general classes $\mathcal{O}_n^{\pm}$, $n > 0$).

THEOREM A. *Let $1 < p \leqslant q < \infty$ and a matrix $(a_{i,j})$ satisfy the condition (14). Then the inequality (11) holds if and only if $A^+ = \max\left\{A_1^+, A_2^+\right\} < \infty$, where*

$$A_1^+ = \sup_{k \geqslant 1} \left(\sum_{n=k}^{\infty} b_{n,k}^q u_n^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} w_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}},$$

$$A_2^+ = \sup_{k \geqslant 1} \left(\sum_{n=k}^{\infty} u_n^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} a_{k,j}^q v_j^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $A^+ \approx C$, where C is the best constant in (11).

THEOREM B. *Let $1 < p \leqslant q < \infty$ and a matrix $(a_{i,j})$ satisfy the condition (15). Then the inequality (11) holds if and only if $A^- = \max\left\{A_1^-, A_2^-\right\} < \infty$, where*

$$A_1^- = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} b_{i,n}^q u_n^q \right)^{\frac{1}{q}} \left(\sum_{j=k}^{\infty} w_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}},$$

$$A_2^- = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} u_n^q \right)^{\frac{1}{q}} \left(\sum_{j=k}^{\infty} a_{j,k}^q v_j^{-p'} \right)^{\frac{1}{p'}}.$$

Moreover, $A^- \approx C$, where C is the best constant in (11).

The main aim of this work is to find new characterizations for the fulfillment of the inequality (11) for matrix operators belonging to the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$. These new characterizations are alternative to those in Theorems A and B, and they have the form of characterizations (6) and (7) given by S. Bloom and R. Kerman.

REMARK 1. Above and hereafter, the notation $M \ll K$ means that there exists a constant α such that $M \leqslant \alpha K$. We write $M \approx K$ if $M \ll K \ll M$.

To prove the two lemmas below, we need the following statement proved in [16].

LEMMA A. *Let $\gamma > 0$ and β_k be a positive sequence. Then for all $j \in \mathbb{N}$*

$$\left(\sum_{k=1}^{j} \beta_k \right)^{\gamma} \approx \sum_{k=1}^{j} \beta_k \left(\sum_{i=1}^{k} \beta_i \right)^{\gamma - 1}, \tag{16}$$

and if $\sum_k \beta_k < \infty$, $1 \leqslant j < N \leqslant \infty$, *then*

$$\left(\sum_{k=j}^{N} \beta_k\right)^{\gamma} \approx \sum_{k=j}^{N} \beta_k \left(\sum_{i=k}^{N} \beta_i\right)^{\gamma-1}. \tag{17}$$

To prove the main results, let us first prove the following two lemmas.

LEMMA 1. *Let* $1 < q < \infty$, $u = \{u_i\}_{i=1}^{\infty}$, $u \geqslant 0$ *and a matrix* $(a_{i,j})$ *of the operator* (9) *belong to the class* $\mathscr{O}_1^+$. *Let for* $f = \{f_i\}_{i=1}^{\infty}$

$$D^+ = \sum_{n=1}^{\infty} \left(u_n \sum_{i=1}^{n} a_{n,i} f_i\right)^q < \infty.$$

Then

$$D^+ \approx D_1^+ + D_2^+, \tag{18}$$

where

$$D_1^+ = \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} w_j f_j\right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} b_{n,i}^{q-1} u_n^q,$$

$$D_2^+ = \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} u_n^q,$$

and the equivalence constants do not depend on f.

Proof. Using (16) and (14) and changing the order of summation, we get

$$D^+ = \sum_{n=1}^{\infty} \left(u_n \sum_{i=1}^{n} a_{n,i} f_i\right)^q \approx \sum_{n=1}^{\infty} u_n^q \sum_{i=1}^{n} a_{n,i} f_i \left(\sum_{j=1}^{i} a_{n,j} f_j\right)^{q-1}$$

$$\approx \sum_{n=1}^{\infty} u_n^q \sum_{i=1}^{n} a_{n,i} b_{n,i}^{q-1} f_i \left(\sum_{j=1}^{i} w_j f_j\right)^{q-1} + \sum_{n=1}^{\infty} u_n^q \sum_{i=1}^{n} a_{n,i} f_i \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^{q-1}$$

$$= \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} w_j f_j\right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} b_{n,i}^{q-1} u_n^q + \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} u_n^q$$

$$= D_1^+ + D_2^+. \quad \square$$

LEMMA 2. *Let* $1 < q < \infty$, $u = \{u_i\}_{i=1}^{\infty}$, $u \geqslant 0$ *and a matrix* $(a_{i,j})$ *of the operator* (13) *belong to the class* $\mathscr{O}_1^-$. *Let for* $f = \{f_i\}_{i=1}^{\infty}$

$$D^- = \sum_{n=1}^{\infty} \left(u_n \sum_{i=n}^{\infty} a_{i,n} f_i\right)^q < \infty.$$

Then

$$D^- \approx D_1^- + D_2^-, \tag{19}$$

where

$$D_1^- = \sum_{i=1}^{\infty} f_i \sum_{j=1}^{i} a_{i,j} u_j^q \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{q-1},$$

$$D_2^- = \sum_{i=1}^{\infty} f_i \left(\sum_{k=i}^{\infty} w_k f_k \right)^{q-1} \sum_{j=1}^{i} a_{i,j} b_{i,j}^{q-1} u_j^q,$$

and the equivalence constants do not depend on f.

Proof. Using (17) and (15) and changing the order of summation, we get

$$D^- = \sum_{n=1}^{\infty} \left(u_n \sum_{i=n}^{\infty} a_{i,n} f_i \right)^q \approx \sum_{n=1}^{\infty} u_n^q \sum_{i=n}^{\infty} a_{i,n} f_i \left(\sum_{k=i}^{\infty} a_{k,n} f_k \right)^{q-1}$$

$$\approx \sum_{n=1}^{\infty} u_n^q \sum_{i=n}^{\infty} a_{i,n} f_i \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{q-1} + \sum_{n=1}^{\infty} u_n^q \sum_{i=n}^{\infty} a_{i,n} b_{i,n}^{q-1} f_i \left(\sum_{k=i}^{\infty} w_k f_k \right)^{q-1}$$

$$= \sum_{i=1}^{\infty} f_i \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{q-1} \sum_{n=1}^{i} a_{i,n} u_n^q + \sum_{i=1}^{\infty} f_i \left(\sum_{k=i}^{\infty} w_k f_k \right)^{q-1} \sum_{n=1}^{i} a_{i,n} b_{i,n}^{q-1} u_n^q$$

$$= D_1^- + D_2^-. \quad \square$$

REMARK 2. In the proofs we assume that $\sum_{i=m}^{n} = 0$ for $m > n$ and $g_0 = 0$.

3. Main results for the class $\mathscr{O}_1^+$

In this Section we present an alternative criterion for the fulfillment of the inequality (11) under the condition of Theorem A.

THEOREM 2. *Let $1 < p \leqslant q < \infty$ and a matrix $(a_{i,j})$ belong to the class $\mathscr{O}_1^+$. Then the inequality (11) holds if and only if $B^+ = \max\left\{B_1^+, B_2^+\right\} < \infty$, where*

$$B_1^+ = \sup_{k \geqslant 1} \left(\sum_{n=k}^{\infty} v_n^{-p'} \left(\sum_{i=n}^{\infty} a_{i,n} b_{i,n}^{q-1} u_i^q \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{n=k}^{\infty} b_{n,k}^q u_n^q \right)^{-\frac{1}{q'}},$$

$$B_2^+ = \sup_{k \geqslant 1} \left(\sum_{n=k}^{\infty} v_n^{-p'} \left(\sum_{i=n}^{\infty} a_{i,n} u_i^q \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{n=k}^{\infty} u_n^q \right)^{-\frac{1}{q'}}.$$

Moreover, $C \approx B^+$, where C is the best constant in (11).

Proof. Necessity. Let the inequality (11) hold. Then the dual inequality

$$\left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n}g_i\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{i=1}^{\infty}\left|u_i^{-1}g_i\right|^{q'}\right)^{\frac{1}{q'}}, \quad g \geqslant 0, \tag{20}$$

also holds. From (18) and from the validity of the inequality (11), we have

$$\sum_{n=1}^{\infty} f_n\left(\sum_{j=1}^{n} w_j f_j\right)^{q-1}\sum_{i=n}^{\infty} a_{i,n}b_{i,n}^{q-1}u_i^q$$

$$\ll \sum_{n=1}^{\infty}\left(u_n\sum_{i=1}^{n} a_{n,i}f_i\right)^q \leqslant C^q\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{q}{p}} < \infty.$$

Hence, due to (14), it follows that $w_n\sum_{i=n}^{\infty} b_{i,n}^q u_i^q \ll \sum_{i=n}^{\infty} a_{i,n}b_{i,n}^{q-1}u_i^q < \infty$ for all $n \geqslant 1$.

If we assume that $g_i = \begin{cases} 0, & 1 \leqslant i \leqslant k-1, \\ b_{i,k}^{q-1}u_i^q, & i \geqslant k, \end{cases}$ for $k > 1$, $k \in \mathbb{N}$, then from (20) we get

$$\left(\sum_{n=k}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n}b_{i,k}^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{i=k}^{\infty} b_{i,k}^q u_i^q\right)^{\frac{1}{q'}}, \quad \forall\, k \in \mathbb{N}.$$

Due to the relation $b_{i,k} \geqslant b_{i,n}$ for $n \geqslant k$, which follows from (14), we obtain

$$\left(\sum_{n=k}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n}b_{i,n}^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{i=k}^{\infty} b_{i,k}^q u_i^q\right)^{\frac{1}{q'}}, \quad \forall\, k \in \mathbb{N},$$

which gives that

$$B_1^+ = \sup_{k\geqslant 1}\left(\sum_{n=k}^{\infty} b_{n.k}^q u_n^q\right)^{-\frac{1}{q'}}\left(\sum_{n=k}^{\infty} v_n^{-p'}\left(\sum_{i=n}^{\infty} a_{i,n}b_{i,n}^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C < \infty. \tag{21}$$

From (18) and from the validity of the inequality (11) we have

$$\sum_{n=1}^{\infty} f_n\left(\sum_{k=1}^{n} a_{n,k}f_k\right)^{q-1}\sum_{i=n}^{\infty} a_{i,n}u_i^q \leqslant \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} a_{n,k}f_k\right)^q$$

$$\leqslant C^q\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{q}{p}} < \infty.$$

Therefore, $\sum_{i=n}^{\infty} a_{i,n} u_i^q < \infty$ for all $n \in \mathbb{N}$. From (14) it follows that $a_{i,n} \geqslant \frac{1}{d} a_{k,n}$ for $i \geqslant k$. Hence,

$$\left(\frac{1}{d} a_{k,n}\right) \sum_{i=n}^{\infty} u_i^q \leqslant \sum_{i=n}^{\infty} a_{i,n} u_i^q < \infty,$$

i.e., $\sum_{i=n}^{\infty} u_i^q < \infty$ for all $n \geqslant 1$.

Now we assume that $g_i = \begin{cases} 0, & 1 \leqslant i \leqslant k-1, \\ u_i^q, & i \geqslant k, \end{cases}$ for $k > 1$, $k \in \mathbb{N}$. Then from (20) we have

$$\left(\sum_{n=k}^{\infty} v_n^{-p'} \left(\sum_{i=n}^{\infty} a_{i,n} u_i^q \right)^{p'} \right)^{\frac{1}{p'}} \leqslant C \left(\sum_{i=k}^{\infty} u_i^q \right)^{\frac{1}{q'}}, \quad \forall\, k \in \mathbb{N},$$

which gives that

$$B_2^+ = \sup_{k \geqslant 1} \left(\sum_{n=k}^{\infty} u_n^q \right)^{-\frac{1}{q'}} \left(\sum_{n=k}^{\infty} v_n^{-p'} \left(\sum_{i=n}^{\infty} a_{i,n} u_i^q \right)^{p'} \right)^{\frac{1}{p'}} \leqslant C < \infty. \tag{22}$$

From (21) and (22) we get

$$B^+ = \max\left\{B_1^+, B_2^+\right\} \leqslant C. \tag{23}$$

Sufficiency. Let $B^+ < \infty$. Using the Hölder inequality, we obtain

$$D_1^+ = \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} w_j f_j \right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} b_{n,i}^{q-1} u_n^q$$

$$\leqslant \|vf\|_p \left(\sum_{i=1}^{\infty} v_i^{-p'} \left(\sum_{j=1}^{i} w_j f_j \right)^{p'(q-1)} \left(\sum_{n=i}^{\infty} a_{n,i} b_{n,i}^{q-1} u_n^q \right)^{p'} \right)^{\frac{1}{p'}} =: J_1^+ \|vf\|_p. \tag{24}$$

Applying the Abel transformation to J_1^+, we find

$$J_1^+ = \left(\sum_{i=1}^{\infty} \sum_{k=i}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} b_{n,k}^{q-1} u_n^q \right)^{p'} \Delta^- \left(\sum_{j=1}^{i} w_j f_j \right)^{p'(q-1)} \right)^{\frac{1}{p'}}, \tag{25}$$

where $\Delta^- F_n = F_n - F_{n-1}$. From $B_1^+ < \infty$ we have

$$\sum_{k=i}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} b_{n,k}^{q-1} u_n^q \right)^{p'} \leqslant (B_1^+)^{p'} \left(\sum_{j=i}^{\infty} b_{j,i}^q u_j^q \right)^{\frac{p'}{q'}}.$$

Applying this relation to (25) and using the notation $\Delta^+ F_n = F_n - F_{n+1}$, we deduce

$$J_1^+ \leqslant B_1^+ \left(\sum_{i=1}^{\infty} \left(\sum_{k=i}^{\infty} b_{k,i}^q u_k^q \right)^{\frac{p'}{q'}} \Delta^- \left(\sum_{j=1}^{i} w_j f_j \right)^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$= B_1^+ \left(\left[\sum_{i=1}^{\infty} \left\{ \sum_{s=i}^{\infty} \Delta^+ \left(\sum_{k=s}^{\infty} b_{k,s}^q u_k^q \right) \left(\Delta^- \left(\sum_{j=1}^{i} w_j f_j \right)^{p'(q-1)} \right)^{\frac{q'}{p'}} \right\}^{\frac{p'}{q'}} \right]^{\frac{q'}{p'}} \right)^{\frac{1}{q'}}$$

(we use the Minkowski inequality)

$$\leqslant B_1^+ \left[\sum_{s=1}^{\infty} \Delta^+ \left(\sum_{k=s}^{\infty} b_{k,s}^q u_k^q \right) \left\{ \sum_{i=1}^{s} \Delta^- \left(\sum_{j=1}^{i} w_j f_j \right)^{p'(q-1)} \right\}^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$= B_1^+ \left[\sum_{s=1}^{\infty} \Delta^+ \left(\sum_{k=s}^{\infty} b_{k,s}^q u_k^q \right) \left(\sum_{j=1}^{s} w_j f_j \right)^q \right]^{\frac{1}{q'}}.$$

Applying the Abel transformation, taking into account that $b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a)$ for $b \geqslant a \geqslant 0$, $\gamma > 0$, and $a_{k,s} \gg b_{k,s} w_s$ for $k \geqslant s$, we obtain

$$D_1^+ \ll B_1^+ \left(\sum_{s=1}^{\infty} f_s \left(\sum_{j=1}^{s} w_j f_j \right)^{q-1} \sum_{k=s}^{\infty} a_{k,s} b_{k,s}^{q-1} u_k^q \right)^{\frac{1}{q'}} \|vf\|_p = B_1^+ (D_1^+)^{\frac{1}{q'}} \|vf\|_p. \quad (26)$$

From (18), (24) and (26), we get

$$D_1^+ \ll B_1^+ (D^+)^{\frac{1}{q'}} \|vf\|_p \ll B^+ (D^+)^{\frac{1}{q'}} \|vf\|_p. \quad (27)$$

Now let us evaluate D_2^+. Using the Hölder inequality, we obtain

$$D_2^+ = \sum_{i=1}^{\infty} f_i \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^{q-1} \sum_{n=i}^{\infty} a_{n,i} u_n^q \leqslant \|vf\|_p J_2^+, \quad (28)$$

where

$$J_2^+ = \left(\sum_{i=1}^{\infty} v_i^{-p'} \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^{p'(q-1)} \left(\sum_{n=i}^{\infty} a_{n,i} u_n^q \right)^{p'} \right)^{\frac{1}{p'}}.$$

We estimate J_2^+ in the same way as J_1^+. Applying the Abel transformation, we find

$$J_2^+ = \left(\sum_{i=1}^{\infty}\sum_{k=i}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{p'} \Delta^-\left(\sum_{j=1}^{i} a_{i,j}f_j\right)^{p'(q-1)}\right)^{\frac{1}{p'}}. \tag{29}$$

From $B_2^+ < \infty$ we have

$$\sum_{k=i}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{p'} \leqslant (B_2^+)^{p'}\left(\sum_{s=i}^{\infty} u_s^q\right)^{\frac{p'}{q'}}.$$

Using this relation and applying the Minkowski inequality to (29), we obtain

$$J_2^+ \leqslant B_2^+\left[\sum_{n=1}^{\infty}\Delta^+\left(\sum_{s=n}^{\infty} u_s^q\right)\left(\sum_{j=1}^{n} a_{n,j}f_j\right)^q\right]^{\frac{1}{q'}}$$

(we use the Abel transformation and $b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a)$ for $b \geqslant a \geqslant 0,\ \gamma > 0$)

$$= B_2^+\left(\sum_{n=1}^{\infty}\Delta^-\left(\sum_{j=1}^{n} a_{n,j}f_j\right)^q \sum_{s=n}^{\infty} u_s^q\right)^{\frac{1}{q'}}$$

$$\approx B_2^+\left(\sum_{n=1}^{\infty} a_{n,n}f_n\left(\sum_{j=1}^{n} a_{n,j}f_j\right)^{q-1} \sum_{s=n}^{\infty} u_s^q\right)^{\frac{1}{q'}}$$

(we take into account that $a_{n,n} \ll a_{s,n}$ for $n \leqslant s$)

$$\ll B_2^+\left(\sum_{n=1}^{\infty} f_n\left(\sum_{j=1}^{n} a_{n,j}f_j\right)^{q-1} \sum_{s=n}^{\infty} a_{s,n}u_s^q\right)^{\frac{1}{q'}}. \tag{30}$$

From (18), (28) and (30) it follows that

$$D_2^+ \ll B_2^+(D_2^+)^{\frac{1}{q'}}\|vf\|_p \ll B_2^+(D^+)^{\frac{1}{q'}}\|vf\|_p \leqslant B^+(D^+)^{\frac{1}{q'}}\|vf\|_p. \tag{31}$$

Then from (18), (27) and (31) we deduce

$$(D^+)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} u_n^q\left(\sum_{i=1}^{n} a_{n,i}f_i\right)^q\right)^{\frac{1}{q}} \ll B^+\|vf\|_p.$$

i.e., the inequality (11) holds with the estimate $C \ll B^+$, where the constant C is the best in (11), which together with (23) gives that $C \approx B$. The proof of Theorem 2 is complete. □

In the proof of Theorem 2, the validity of inequality (11) is established on the basis of the validity of the dual inequality (20). Thus, from Theorem 2 we obtain the following statement.

THEOREM 3. *Let* $1 < p \leqslant q < \infty$ *and a matrix* $(a_{i,j})$ *belong to the class* $\mathscr{O}_1^+$. *Then the inequality* (12) *holds if and only if* $(B^+)^* = \max\left\{(B_1^+)^*, (B_2^+)^*\right\} < \infty$, *where*

$$(B_1^+)^* = \sup_{k\geqslant 1}\left(\sum_{n=k}^{\infty} u_n^q \left(\sum_{i=n}^{\infty} a_{i,n} b_{i,n}^{p'-1} v_i^{-p'}\right)^q\right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} b_{i,k}^q v_i^{-p'}\right)^{-\frac{1}{p}},$$

$$(B_2^+)^* = \sup_{k\geqslant 1}\left(\sum_{n=k}^{\infty} u_n^q \left(\sum_{i=n}^{\infty} a_{i,n} v_i^{-p'}\right)^q\right)^{\frac{1}{q}} \left(\sum_{i=k}^{\infty} v_i^{-p'}\right)^{-\frac{1}{p}}.$$

Moreover, $C \approx (B^+)^*$, *where* C *is the best constant in* (12).

4. Main results for the class $\mathscr{O}_1^-$

In this Section we consider the inequality (12) for the matrix operator (13) from the class $\mathscr{O}_1^-$. The validity of the inequality (11) for this class follows from the validity of the inequality (12) as a corollary.

THEOREM 4. *Let* $1 < p \leqslant q < \infty$ *and a matrix* $(a_{i,j})$ *belong to the class* $\mathscr{O}_1^-$. *Then the inequality* (12) *holds if and only if* $(B^-)^* = \max\left\{(B_1^-)^*, (B_2^-)^*\right\} < \infty$, *where*

$$(B_1^-)^* = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_k^{-p'} \left(\sum_{i=1}^{n} a_{n,i} u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n=1}^{k} u_n^q\right)^{-\frac{1}{q'}},$$

$$(B_2^-)^* = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_k^{-p'} \left(\sum_{i=1}^{n} a_{n,i} b_{n,i}^{q-1} u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n=k}^{\infty} b_{n,k}^q u_n^q\right)^{-\frac{1}{q'}}.$$

Moreover, $C \approx (B^-)^*$, *where* C *is the best constant in* (12).

Proof. Necessity. Let the inequality (12) hold. Then the dual inequality

$$\left(\sum_{n=1}^{\infty} v_n^{-p'} \left(\sum_{i=1}^{n} a_{n,i} g_i\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{n=1}^{\infty} \left|u_n^{-1} g_n\right|^{q'}\right)^{\frac{1}{q'}}, \quad g \geqslant 0, \tag{32}$$

also holds. From the validity of the inequality (12) it follows that $\sum_{i=1}^{n} u_i^q < \infty$ for all $n \in \mathbb{N}$.

If we assume that $g_i = \begin{cases} u_i^q, & 1 \leqslant i \leqslant k, \\ 0, & i > k, \end{cases}$ for all $k \in \mathbb{N}$, then from (32) we have

$$\left(\sum_{n=1}^{k} v_n^{-p'} \left(\sum_{i=1}^{n} a_{n,i} u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{n=1}^{k} u_n^q\right)^{\frac{1}{q'}}, \quad \forall\, k \in \mathbb{N},$$

which yields that

$$(B_1^-)^* = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'}\left(\sum_{i=1}^{n} a_{n,i}u_i^q\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{k} u_n^q\right)^{-\frac{1}{q'}} \leqslant C < \infty. \tag{33}$$

From (19) and from the validity of the inequality (12), we get

$$\sum_{i=1}^{\infty} f_i\left(\sum_{k=i}^{\infty} w_k f_k\right)^{q-1}\sum_{j=1}^{i} a_{i,j}b_{i,j}^{q-1}u_j^q \leqslant C^q\left(\sum_{i=1}^{\infty}|v_i f_i|^p\right)^{\frac{q}{p}} < \infty.$$

Due to the condition $a_{i,j} \gg b_{i,j}^{q-1}w_i$, the latter gives that $w_i\sum_{j=1}^{i} b_{i,j}^q u_j^q \ll \sum_{j=1}^{i} a_{i,j}b_{i,j}^{q-1}u_j^q < \infty$ for all $i \geqslant 1$.

Now we assume that $g_i = \begin{cases} b_{k,i}^{q-1}u_i^q, & 1\leqslant i\leqslant k,\\ 0, & i>k,\end{cases}$ for all $k\in\mathbb{N}$. Then from (32) we deduce

$$\left(\sum_{n=1}^{k} v_n^{-p'}\left(\sum_{i=1}^{n} a_{n,i}b_{k,i}^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{i=1}^{k} b_{k,i}^q u_i^q\right)^{\frac{1}{q'}}, \quad \forall\, k\in\mathbb{N}.$$

Due to of the relation $b_{k,i} \geqslant b_{n,i}$ for $k \geqslant n$, which follows from (15), we obtain

$$(B_2^-)^* = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'}\left(\sum_{i=1}^{n} a_{n,i}b_{n,i}^{q-1}u_i^q\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{k} b_{k,n}^q u_n^q\right)^{-\frac{1}{q'}} \leqslant C < \infty. \tag{34}$$

From (33) and (34) we have

$$(B^-)^* = \max\left\{(B_1^-)^*, (B_2^-)\right\} \leqslant C. \tag{35}$$

Sufficiency. Let $(B^-)^* < \infty$. Using the Hölder inequality, we obtain

$$D_1^- = \sum_{i=1}^{\infty} f_i \sum_{j=1}^{i} a_{i,j}u_j^q\left(\sum_{k=i}^{\infty} a_{k,i}f_k\right)^{q-1}$$

$$\leqslant \|vf\|_p\left(\sum_{i=1}^{\infty} v_i^{-p'}\left(\sum_{j=1}^{i} a_{i,j}u_j^q\right)^{p'}\left(\sum_{k=i}^{\infty} a_{k,i}f_k\right)^{p'(q-1)}\right)^{\frac{1}{p'}} =: J_1^-\|vf\|_p. \tag{36}$$

Let us estimate J_1^-. Applying the Abel transformation, we find

$$J_1^- = \left(\sum_{i=1}^{\infty}\sum_{k=1}^{i} v_k^{-p'}\left(\sum_{j=1}^{k} a_{k,j}u_j^q\right)^{p'}\Delta^+\left(\sum_{k=i}^{\infty} a_{k,i}f_k\right)^{p'(q-1)}\right)^{\frac{1}{p'}}.$$

Using $(B_1^-)^* < \infty$, we get

$$J_1^- \leqslant (B_1^-)^* \left(\sum_{i=1}^{\infty} \left(\sum_{n=1}^{i} u_n^q \right)^{\frac{p'}{q'}} \Delta^+ \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$= (B_1^-)^* \left(\left[\sum_{i=1}^{\infty} \left\{ \sum_{s=1}^{i} \Delta^- \left(\sum_{n=1}^{s} u_n^q \right) \left(\Delta^+ \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{p'(q-1)} \right)^{\frac{q'}{p'}} \right\}^{\frac{p'}{q'}} \right]^{\frac{q'}{p'}} \right)^{\frac{1}{q'}}$$

(we use the Minkowski inequality)

$$\leqslant (B_1^-)^* \left[\sum_{s=1}^{\infty} \Delta^- \left(\sum_{n=1}^{s} u_n^q \right) \left\{ \sum_{i=s}^{\infty} \Delta^+ \left(\sum_{k=i}^{\infty} a_{k,i} f_k \right)^{p'(q-1)} \right\}^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$= (B_1^-)^* \left[\sum_{s=1}^{\infty} \Delta^- \left(\sum_{n=1}^{s} u_n^q \right) \left(\sum_{k=s}^{\infty} a_{k,s} f_k \right)^q \right]^{\frac{1}{q'}}$$

(we apply the Abel transformation and $b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a)$ for $b \geqslant a \geqslant 0,\ \gamma > 0$)

$$\leqslant (B_1^-)^* \left[\sum_{s=1}^{\infty} \Delta^+ \left(\sum_{k=s}^{\infty} a_{k,s} f_k \right)^q \sum_{n=1}^{s} u_n^q \right]^{\frac{1}{q'}}$$

$$\approx (B_1^-)^* \left[\sum_{s=1}^{\infty} a_{s,s} f_s \left(\sum_{k=s}^{\infty} a_{k,s} f_k \right)^{q-1} \sum_{n=1}^{s} u_n^q \right]^{\frac{1}{q'}}$$

(we take into account that $a_{s,s} \ll a_{s,n}$ for $s \geqslant n$)

$$\leqslant (B_1^-)^* \left[\sum_{s=1}^{\infty} f_s \left(\sum_{k=s}^{\infty} a_{k,s} f_k \right)^{q-1} \sum_{n=1}^{s} a_{s,n} u_n^q \right]^{\frac{1}{q'}}. \tag{37}$$

From (19), (36) and (37), we obtain

$$D_1^- \ll (B_1^-)^* (D_1^-)^{\frac{1}{q'}} \|vf\|_p \ll (B_1^-)^* (D^-)^{\frac{1}{q'}} \|vf\|_p. \tag{38}$$

Now we estimate D_2^-. Using the Hölder inequality, we find

$$D_2^- = \sum_{i=1}^{\infty} f_i \left(\sum_{k=i}^{\infty} w_k f_k \right)^{q-1} \sum_{j=1}^{i} a_{i,j} b_{i,j}^{q-1} u_j^q \leqslant \|vf\|_p J_2^-, \tag{39}$$

where

$$J_2^- = \left(\sum_{i=1}^{\infty} v_i^{-p'} \left(\sum_{k=i}^{\infty} w_k f_k\right)^{p'(q-1)} \left(\sum_{j=1}^{i} a_{i,j} b_{i,j}^{q-1} u_j^q\right)^{p'}\right)^{\frac{1}{p'}}.$$

Let us estimate the value of J_2^- in the same way as J_1^-.

$$J_2^- = \left(\sum_{i=1}^{\infty}\sum_{k=1}^{i} v_k^{p'} \left(\sum_{j=1}^{k} a_{k,j} b_{k,j}^{q-1} u_j^q\right)^{p'} \Delta^+ \left(\sum_{k=i}^{\infty} w_k f_k\right)^{p'(q-1)}\right)^{\frac{1}{p'}}$$

$$\leqslant (B_2^-)^* \left[\sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} b_{i,k}^q u_k^q\right)^{\frac{p'}{q'}} \Delta^+ \left(\sum_{k=i}^{\infty} w_k f_k\right)^{p'(q-1)}\right]^{\frac{1}{p'}}$$

$$= (B_2^-)^* \left(\left[\sum_{i=1}^{\infty} \left\{\sum_{s=1}^{i} \Delta^- \left(\sum_{k=1}^{s} b_{s,k}^q u_k^q\right) \left(\Delta^+ \left(\sum_{k=i}^{\infty} w_k f_k\right)^{p'(q-1)}\right)^{\frac{q'}{p'}}\right\}^{\frac{p'}{q'}}\right]^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}$$

(we use the Minkowski inequality)

$$= (B_2^-)^* \left(\sum_{s=1}^{\infty} \Delta^- \left(\sum_{k=1}^{s} b_{s,k}^q u_k^q\right) \left(\sum_{k=s}^{\infty} w_k f_k\right)^q\right)^{\frac{1}{q'}}$$

(we apply the Abel transformation and $b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a)$ for $b \geqslant a \geqslant 0,\ \gamma > 0$)

$$= (B_2^-)^* \left(\sum_{n=1}^{\infty} \Delta^+ \left(\sum_{k=s}^{\infty} w_k f_k\right)^q \sum_{k=1}^{s} b_{s,k}^q u_k^q\right)^{\frac{1}{q'}}$$

$$\approx (B_2^-)^* \left(\sum_{s=1}^{\infty} w_s f_s \left(\sum_{k=s}^{\infty} w_k f_k\right)^{q-1} \sum_{k=1}^{s} b_{s,k}^q u_k^q\right)^{\frac{1}{q'}}$$

(we take into account that $b_{s,k} w_s \ll a_{s,k}$ for $s \geqslant k$)

$$\ll (B_2^-)^* \left(\sum_{s=1}^{\infty} f_s \left(\sum_{k=s}^{\infty} w_k f_k\right)^{q-1} \sum_{k=1}^{s} a_{s,k} b_{s,k}^{q-1} u_k^q\right)^{\frac{1}{q'}} \ll (B_2^-)^* (D_2^-)^{\frac{1}{q'}}. \qquad (40)$$

From (19), (39) and (40) it follows that

$$D_2^- \ll (B_1^-)(D_2^-)^{\frac{1}{q'}} \|vf\|_p \ll (B_1^-)(D^-)^{\frac{1}{q'}} \|vf\|_p. \qquad (41)$$

Thus, from (19), (38) and (41) we deduce

$$(D^-)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{i=n}^{\infty} a_{i,n} f_i\right)^q\right)^{\frac{1}{q}} \ll (B^-)^* \|vf\|_p,$$

i.e., the inequality (12) holds with the estimate $C \ll (B^-)^*$, where C is the best constant in (12), which together with (35) gives that $C \approx (B^-)^*$. The proof of Theorem 4 is complete. □

In the proof of Theorem 4, the validity of inequality (12) is established on the basis of the validity of the dual inequality (32). The inequality (32) is an inequality of the form (11), then from Theorem 4 we obtain the following statement, which is an alternative criterion for the fulfillment of the inequality (11) under the conditions of Theorem B.

THEOREM 5. *Let $1 < p \leqslant q < \infty$ and a matrix $(a_{i,j})$ belong to the class $\mathcal{O}_1^-$. Then the inequality (11) holds if and only if $B^- = \max\left\{B_1^-, B_2^-\right\} < \infty$, where*

$$B_1^- = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} u_n^q \left(\sum_{i=1}^{n} a_{n,i} v_i^{-p'}\right)^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{k} v_n^{-p'}\right)^{-\frac{1}{p}},$$

$$B_2^- = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} u_n^q \left(\sum_{i=1}^{n} a_{n,i} b_{n,i}^{p'-1} v_i^{-p'}\right)^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{k} b_{k,i}^q v_i^{-p'}\right)^{-\frac{1}{p}}.$$

Moreover, $C \approx B^-$, where C is the best constant in (11).

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