

GLOBAL ATTRACTORS AND ASYMPTOTIC GAIN PROPERTY FOR NON-AUTONOMOUS INCLUSION OF REACTION-DIFFUSION TYPE

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Abstract. We investigate global resolvability and stability of attractors for parabolic inclusion with multi-valued interaction function of reaction-diffusion type and non-autonomous disturbances. For the class of L^2 -disturbances, we prove existence of global solutions in the phase space L^2 . In the class of translation-bounded disturbances we prove that obtained global solutions generate the family of multi-valued semiprocesses which possesses a uniform attractor. Finally, for L^∞ -disturbances we show that the global attractor of unperturbed system is stable w.r.t. disturbances in the asymptotic gain sense.

Key words: parabolic inclusion, reaction-diffusion, uniform attractor, stability, asymptotic gain.

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1. Introduction

Global attractors play an important role in the qualitative theory of dissipative infinite-dimensional dynamical systems [1, 2]. In recent years, many works have appeared in which the classical theory of global attractors was extended to problems with non-autonomous, stochastic, impulsive, and multi-valued perturbations [3–9]. For evolutionary inclusions, such results were first obtained by V.S. Melnik and J. Valero, who introduced the concept of a multi-valued semiflow generated by integral solutions of inclusions of subdifferential type [10]. In subsequent works, the topological and metric properties of attractors of dissipative inclusions, their structure and stability in the sense of Lyapunov were investigated [11–13]. For inclusions with multi-valued non-autonomous right-hand sides of linear growth in [14, 15], the results on existence of uniform attractors of the corresponding semiprocesses were obtained. In the present work, we investigate an evolutionary inclusion with a multi-valued interaction function of the polynomial

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growth in the presence of non-autonomous perturbations. The global solvability in the phase space L^2 and the existence of a uniform attractor for the corresponding family of multivalued semiprocesses is proved. Using the general approach developed in [16–19], the robust stability of the global attractor of an unperturbed system with respect to the magnitude of non-autonomous perturbations is shown. The work is structured as follows: in the second section, the problem statement and the necessary properties regarding the weak solvability of reaction-diffusion type equations are given. In the third section, the general global resolvability result in the phase space L^2 is proved. In the fourth section, under the condition of translation-bounded perturbations, the existence of uniform attractors of semiprocesses generated by weak solutions of the original problem is proved and their dependence on the parameter is investigated. In the final section, in the class of bounded perturbations, a robust estimate of the AG type is proved, which characterizes the deviation of the trajectories of the perturbed problem from the global attractor.

2. Setting of the problem

In the bounded domain $\Omega \subset \mathbb{R}^m$, $m \geq 1$, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \in f(u) + d(t, x), & (t, x) \in Q = (0, +\infty) \times \Omega, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \quad (2.1)$$

where $d \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ is a non-autonomous disturbance function, $u_0 \in L^2(\Omega)$, and the multi-valued function f satisfies the following assumptions:

$$\begin{aligned} f : \mathbb{R} &\rightarrow C_v(\mathbb{R}) \text{ is upper semicontinuous,} \\ &\exists C > 0, \alpha_1, \alpha_2 > 0, p \geq 2, \text{ such that} \\ \forall s \in \mathbb{R}, \forall \xi \in f(s) &\quad -C - \alpha_1|s|^p \leq \xi \cdot s \leq C - \alpha_2|s|^p. \end{aligned} \quad (2.2)$$

Here $C_v(\mathbb{R})$ denotes the set of all non-empty, convex, compact sets in \mathbb{R} , upper semicontinuity is understood in the standard sense [20].

Our first goal is to prove global resolvability of (2.1) for every $u_0 \in L^2(\Omega)$ in the sense of the following definition.

Definition 2.1. A function $u = u(t, x)$, $(t, x) \in Q_T = (0, T) \times \Omega$ is called a (weak) solution of (2.1) on $(0, T)$, if $u \in L^2(0, T; H^1_0(\Omega))$ and there exists $l \in L^q(Q_T)$, $\frac{1}{q} + \frac{1}{p} = 1$ such that $\forall v \in C^\infty_0(\Omega)$, $\forall \eta \in C^\infty_0(0, T)$

$$\begin{aligned} - \int_0^T (u(t), v) \eta_t dt + \int_0^T (\nabla u(t), \nabla v) \eta dt \\ = \int_0^T \int_\Omega (l(t, x) + d(t, x)) v(x) \eta(t) dx dt, \end{aligned} \quad (2.3)$$

$$l(t, x) \in f(u(t, x)) \text{ almost everywhere (a.e.) on } Q_T \quad (2.4)$$

and the initial condition holds.

Here by $\|\cdot\|$ and (\cdot, \cdot) we denote the norm and scalar product in the phase space $X = L^2(\Omega)$.

We note that equality (2.3) means [2] that

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega) + L^q(Q_T)) \subset L^q(0, T; H^{-s}(\Omega)),$$

where

$$s \geq \max \left\{ 1, m \left(\frac{1}{2} - \frac{1}{p} \right) \right\}.$$

Therefore, due to [2], $u \in \mathbb{C}([0, T]; H^{-s}(\Omega))$, and the initial condition $u|_{t=0} = u_0$ makes sense.

Our second goal is to show that under additional translation-bounded assumption

$$\sup_{t \geq 0} \int_t^{t+1} \|d(s)\|^2 ds < \infty,$$

obtained solutions generate the family of multi-valued semiprocesses $\{U_\sigma\}_{\sigma \in \Sigma(d)}$, which possesses uniform attractor $\Theta_{\Sigma(d)}$.

Here $U_\sigma(t, \tau, u_\tau)$ denotes the set of solutions of (2.1) with disturbances σ at the moment $t \geq \tau$ with initial data $u|_{t=\tau} = u_\tau$, and $\sigma = \sigma(t, x)$ belongs to the shift-invariant space $\Sigma(d)$.

Assuming that $d \in L^\infty(\mathbb{R}_+; L^2(\Omega))$, we finally prove that

$$\text{dist}(\Theta_{\Sigma(d)}, \Theta) \rightarrow 0 \text{ as } \|d\|_\infty \rightarrow 0, \quad (2.5)$$

where Θ is the global attractor of the unperturbed system ($d \equiv 0$).

It allows us to show the following robust estimate: $\forall u_0 \in L^2(\Omega)$

$$\overline{\lim}_{t \rightarrow \infty} \|U_d(t, 0, u_0)\|_\Theta \leq \gamma(\|d\|_\infty), \quad (2.6)$$

where γ belongs to the comparison class K - class of continuous, monotonically increasing functions from \mathbb{R}_+ to \mathbb{R}_+ with $\gamma(0) = 0$ [21],

$$\|d\|_\infty := \text{ess sup}_{t \geq 0} \|d(t)\|,$$

$$\text{for } A \subset X \quad \|A\|_\Theta = \text{dist}(A, \Theta) = \sup_{a \in A} \inf_{\theta \in \Theta} \|a - \theta\|.$$

Now we discuss some results concerning problem (2.1), which are already known. If f is a single-valued continuous function satisfying (2.2) then Definition 2.1 coincides with the definition of classical weak solution for reaction-diffusion equation [2, 12]. But multi-valued upper semicontinuous function may have no continuous selector [20], which means that we cannot use directly resolvability

results for reaction-diffusion equation. Recently, in [22] it has been proven the existence of weak solution for (2.1) under condition (2.2) in the autonomous case $d(t, x) \equiv d(x)$.

Note that if u is a weak solution of (2.1) on $(0, T)$ and, additionally, $u \in L^p(Q_T)$, then due to [4] the function $[0, T] \ni t \mapsto u(t) \in L^2(\Omega)$ is absolutely continuous and for a.e. $t \in [0, T)$

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|\nabla u(t)\|^2 = \int_{\Omega} (l(t, x) + d(t, x)) u(t, x) dx. \quad (2.7)$$

If $d(t, x) \equiv d(x)$ then (2.7) implies dissipation and asymptotic compactness of the multi-valued semiflow $G : \mathbb{R}_+ \times X \rightarrow 2^X$ (see Remark 4.1 below) generated by weak solutions of (2.1) belonging to $L^p_{loc}(0, +\infty; L^p(\Omega))$ [2]. It means that the m -semiflow G possesses a global attractor, i.e., there exists a compact set $\Theta \subset X$ such that

$$\forall t > 0 \quad G(t, \Theta) = \Theta;$$

$$\forall r > 0 \quad \sup_{\|u_0\| \leq r} \text{dist}(G(t, u_0), \Theta) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In the non-autonomous case it is known [14] that under translation-bounded assumption on d and linear growth assumption on f the problem (2.1) is globally resolvable (in this case $p = q = 2$) in X and its weak solutions generate the family of semiprocesses with uniform attractor. In the present paper we extend these results to the case $p \geq 2$.

Robust stability of global attractors with respect to non-autonomous L^∞ -disturbances has been studied in [16, 17] both for parabolic and hyperbolic evolutionary problems, including non-uniqueness case [18, 19]. Taking into account the global Lyapunov stability of the global attractor [12], the desired result would be to obtain the following robust estimate (Input to State Stability, ISS [23]):

$$\forall u_0 \in X, \quad \forall t \geq 0$$

$$\|U_d(t, 0, u_0)\|_{\Theta} \leq \beta(\|u_0\|_{\Theta}, t) + \gamma(\|d\|_{\infty}),$$

where β belongs to the class of comparison functions \mathcal{KL} [21].

Unfortunately, this property is not guaranteed even in the case when Θ consists of a single point [24]. However, we can establish (2.6) – a weaker variant of the last inequality called the asymptotic gain (AG) property. For this purpose, we use the uniform attractors technique, in particular, the limit equality (2.5). Previously, this approach was used to establish the AG stability of the global attractor for dissipative evolutionary equations [25]. The novelty of the present paper is the proof of such an estimate, specifically for evolutionary inclusions.

3. Global resolvability of (2.1)

We make use of the following result.

Lemma 3.1. *Assume that $d(t, x) \equiv d(x) \in L^2(\Omega)$ and conditions (2.2) hold true. Then for every $T > 0$, $u_0 \in L^2(\Omega)$ there exists at least one solution of (2.1) on $(0, T)$ in the sense of Definition 2.1, which belongs to $L^p(Q_T)$.*

Additionally to this fact, we can observe that in the class $L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T)$ equality (2.3) is equivalent to the following: $\forall v \in C_0^\infty(\Omega)$ the function $t \mapsto (u(t), v)$ is absolutely continuous, and a.e. on $(0, T)$:

$$\frac{d}{dt}(u(t), v) + (\nabla u(t), \nabla v) = \int_{\Omega} (l(t, x) + d(t, x)) v(x) dx. \quad (3.1)$$

Indeed, let $u \in L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T)$ satisfy (2.3) with some function $l \in L^q(Q_T)$. Then according to [2]

$$\frac{\partial u}{\partial t} - \Delta u - l - d = 0 \quad \text{in } L^q(0, T; H^{-s}(\Omega)).$$

Due to [1], Lemma 3.1 it means that $\forall v \in H^s(\Omega)$

$$\left\langle \frac{\partial u}{\partial t} - \Delta u - l - d, v \right\rangle = 0 \quad \text{a.e. on } (0, T),$$

or, equivalently,

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle = (\nabla u, \nabla v) + \int_{\Omega} (l(t, x) + d(t, x)) v(x) dx.$$

Moreover, from the inclusion $\frac{\partial u}{\partial t} \in (L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T))^*$ we deduce that $\forall v \in H^s(\Omega)$ the function $t \mapsto (u(t), v)$ is absolutely continuous and

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle = \frac{d}{dt}(u(t), v) \quad \text{a.e.}$$

And vice versa, if for $u \in L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T)$ and for $v \in C_0^\infty(\Omega)$ the function $t \mapsto (u(t), v)$ is absolutely continuous and (3.1) takes place, then after multiplying by $\eta \in C_0^\infty(0, T)$ and integrating by parts we obtain (2.3).

Now we are in position to formulate our resolvability result.

Theorem 3.1. *Assume that condition (2.2) holds. Then for every $T > 0$, $d \in L^2(Q_T)$, $u_0 \in L^2(\Omega)$ problem (2.1) has at least one solution on $(0, T)$, which belongs to $L^p(Q_T)$, and, moreover, for every such solution the following inequality holds for a.e. $t \in (0, T)$*

$$\frac{d}{dt} \|u(t)\|^2 + \delta (\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{L^p}^p) \leq C_1 (1 + \|d(t)\|^2), \quad (3.2)$$

where positive constants δ, C_1 do not depend on T, d, u_0 .

Proof. It is known [26] that for $d \in L^2(Q_T)$ there exists a sequence of step functions $\{d^N\}_{N=1}^\infty$ such that $d^N \rightarrow d$ in $L^2(Q_T)$, where

$$d^N(t, x) = \sum_{i=1}^N d_i^{(N)}(x) \chi_{[t_{i-1}^N, t_i^N]}(t),$$

$$d_i^{(N)} \in L^2(\Omega), \quad \chi_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}, \quad 0 = t_0^N < t_1^N < \dots < t_N^N = T.$$

We are going to prove that for every $N \geq 1$ problem (2.1) with non-autonomous term $d^N(t, x) = \sum_{i=1}^N d_i(x) \chi_{[t_{i-1}, t_i]}(t)$ (we omit index N over d_i and t_i for now) has a solution u^N on $(0, T)$.

Indeed, due to Lemma 3.1 there exists $u_1 = u_1(t, x)$, $(t, x) \in Q_{(0, t_1)} = (0, t_1) \times \Omega$ – solution of (2.1) on $(0, t_1)$, which belongs to $\mathbb{C}([0, t_1]; L^2(\Omega)) \cap L^p(Q_{(0, t_1)})$ and satisfies (3.2) with $d(t, x) \equiv d_1(x)$. Moreover, $\forall v \in C_0^\infty(\Omega)$ $t \mapsto (u_1(t), v)$ is absolutely continuous on $[0, t_1]$ and satisfies (3.1) a.e. on $(0, t_1)$ with $d \equiv d_1(x)$, $l = l_1(t, x)$,

$$l_1(t, x) \in f(u_1(t, x)) \text{ a.e. on } Q_{(0, t_1)}.$$

After that we consider (2.1) on $Q_{(t_1, t_2)} = (t_1, t_2) \times \Omega$ with initial data $u|_{t=t_1} = u_1(t_1)$. Applying Lemma 3.1 again, we get $u_2 = u_2(t, x)$, $(t, x) \in Q_{(t_1, t_2)}$ – solution of (2.1) on (t_1, t_2) , $u_2|_{t=t_1} = u_1(t_1)$, $u_2 \in \mathbb{C}([t_1, t_2]; L^2(\Omega)) \cap L^p(Q_{(t_1, t_2)})$, u_2 satisfies (3.2) with $d(t, x) = d_2(x)$, $t \mapsto (u_2(t), v)$ is absolutely continuous on $[t_1, t_2]$ and satisfies (3.1) a.e. on (t_1, t_2) with $d \equiv d_2(x)$, $l = l_2(t, x)$,

$$l_2(t, x) \in f(u_2(t, x)) \text{ a.e. on } Q_{(t_1, t_2)}.$$

Repeating this procedure for $(t_2, t_3), \dots, (t_{N-1}, T)$, we have that the function

$$u^N(t, x) = \sum_{i=1}^N u_i(t, x) \chi_{[t_{i-1}, t_i]}(t), \quad (t, x) \in Q_T,$$

belongs to the class $L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T)$, $\forall v \in C_0^\infty(\Omega)$ $t \mapsto (u^N(t), v)$ is absolutely continuous on $[0, T]$, u^N satisfies (3.1) a.e. on $(0, T)$ with $d \equiv d^N(t, x)$, $l = l^N(t, x) = \sum_{i=1}^N l_i(t, x) \chi_{[t_{i-1}, t_i]}(t)$, and

$$l^N(t, x) \in f(u^N(t, x)) \text{ a.e. on } Q_T.$$

Therefore, u^N is a solution of (2.1) on $(0, T)$ with the corresponding function l^N , and u^N satisfies (3.2) with d^N .

As $d^N \rightarrow d$ in $L^2(Q_T)$, so (2.2), (3.2) yield that

$$\{u^N\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T), \quad (3.3)$$

$$\{l^N\} \text{ is bounded in } L^q(Q_T).$$

So, up to subsequence

$$u^N \rightharpoonup u \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \cap L^p(Q_T), \quad (3.4)$$

$$l^N \rightharpoonup l \text{ weakly in } L^q(Q_T).$$

Then we can pass to the limit in equality (2.3) and deduce that functions u and l satisfy (2.3) and, moreover, (3.3) implies that

$$\left\{ \frac{\partial u^N}{\partial t} \right\} \text{ is bounded in } L^q(0, T; H^{-s}(\Omega)). \quad (3.5)$$

Properties (3.3), (3.5) and the compactness embedding theorem [2] guarantee that up to subsequence

$$u^N \rightarrow u \text{ in } L^2(Q_T). \quad (3.6)$$

In particular, $u^N(t, x) \rightarrow u(t, x)$ a.e. on Q_T .

From Mazur's theorem we can find \tilde{l}^N - convex combinations of $\{l^i\}_{i=1}^N$ such that

$$\tilde{l}^N \rightarrow l \text{ in } L^q(Q_T) \text{ and a.e. on } Q_T. \quad (3.7)$$

Upper semicontinuity of $f : \mathbb{R} \rightarrow C_v(\mathbb{R})$ implies that for a.e. t, x and for all $\varepsilon > 0$ there exists $n_0 = n_0(t, x)$ s.t. $\forall N \geq n_0$

$$l^N(t, x) \in f(u^N(t, x)) \subset O_\varepsilon(f(u(t, x))),$$

which means that

$$\tilde{l}^N(t, x) \in O_\varepsilon(f(u(t, x))).$$

Then pointwise convergence (3.7) and arbitrary choice of ε complete the proof, that u is a solution of (2.1), which belongs to $L^p(Q_T)$. Inequality (3.2) is a direct consequence of equality (2.7).

Theorem is proved. \square

Corollary 3.1. *We can reformulate Definition 2.1 and speak about solution of (2.1) on (τ, T) . Then Theorem 3.1 allows us to claim that for every $\tau \geq 0$, $u_\tau \in L^2(\Omega)$ problem (2.1) has a solution on $(\tau, +\infty)$ with initial data $u|_{t=\tau} = u_\tau$, which belongs to $L_{loc}^p(\tau, +\infty; L^p(\Omega))$, i.e., there exists a function $u \in L_{loc}^2(\tau, +\infty; H_0^1(\Omega)) \cap L_{loc}^p(\tau, +\infty; L^p(\Omega)) \cap \mathbb{C}([\tau, +\infty); L^2(\Omega))$ with $u|_{t=\tau} = u_\tau$, which is a solution of (2.1) on $(\tau, T) \forall T > \tau$.*

Therefore, for every $t \geq \tau \geq 0$, $u_\tau \in L^2(\Omega)$ we can correctly define the following set

$$U_d(t, \tau, u_\tau) = \{u(t) \mid u(\cdot) \text{ is a solution of (2.1) on } (\tau, \infty) \quad (3.8)$$

$$\text{such that } u(\tau) = u_\tau \text{ and } u \in L_{loc}^p(\tau, +\infty; L^p(\Omega))\}.$$

The index d in (3.8) means that we consider problem (2.1) with disturbance function $d(t, x)$.

In the next section we will discuss some dynamical properties of (3.8).

4. Uniform attractors for semiprocesses, generated by (2.1)

Definition 4.1. [18] Let X be a Banach space, Σ be a shift-invariant metric space, $\mathcal{R} = \{(t, \tau) \in \mathbb{R}_+^2 \mid t \geq \tau \geq 0\}$. Then the family of maps $\{U_\sigma : \mathcal{R} \times X \rightarrow 2^X\}_{\sigma \in \Sigma}$ is called a multi-valued semiprocesses family (m-semiprocesses family for short) if:

$$\begin{aligned} \forall \sigma \in \Sigma \quad \forall u \in X \quad \forall t \geq s \geq \tau \geq 0 \quad \forall h \geq 0 \\ U_\sigma(\tau, \tau, u) = u, \quad U_\sigma(t, \tau, u) \subset U_\sigma(t, s, U_\sigma(s, \tau, u)), \\ U_\sigma(t+h, \tau+h, u) \subset U_{\sigma(\cdot+h)}(t, \tau, u). \end{aligned}$$

Definition 4.2. [18] A compact set $\Theta_\Sigma \subset X$ is called a uniform attractor for m-semiprocesses family $\{U_\sigma\}_{\sigma \in \Sigma}$ if for every bounded set $B \subset X$

$$\text{dist}\left(\bigcup_{\sigma \in \Sigma} U_\sigma(t, 0, B), \Theta_\Sigma\right) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (4.1)$$

and Θ_Σ is the minimal among all such sets.

Lemma 4.1. [18] Assume that a m-semiprocesses family $\{U_\sigma\}_{\sigma \in \Sigma}$ satisfies the following conditions:

1) there exists bounded set $B_0 \subset X$ such that for every bounded set $B \subset X$ there exists $T = T(B) > 0$ such that

$$\forall t \geq T(B) \quad \forall \sigma \in \Sigma \quad U_\sigma(t, 0, B) \subset B_0; \quad (4.2)$$

2) for every $\{\sigma_n\} \subset \Sigma$, $\{t_n \nearrow \infty\}$, and bounded $\{u_n\} \subset X$ every sequence $\{\xi_n \in U_{\sigma_n}(t_n, 0, u_n)\}$ is precompact in X .

Then $\{U_\sigma\}_{\sigma \in \Sigma}$ possesses uniform attractor Θ_Σ .

3) If, additionally, the multi-valued map

$$\Sigma \times X \ni (\sigma, u) \mapsto U_\sigma(t, 0, u)$$

has closed graph, then

$$\forall t \geq 0 \quad \Theta_\Sigma \subset \bigcup_{\sigma \in \Sigma} U_\sigma(t, 0, \Theta_\Sigma). \quad (4.3)$$

Remark 4.1. If $\Sigma = \{0\}$ (unperturbed case), then $G(t, u) := U_0(t, 0, u)$ is m-semiflow [10], i.e., $\forall u \in X, \forall t, s \geq 0$

$$G(0, u) = u, \quad G(t+s, u) \subset G(t, G(s, u)),$$

and $\Theta_{\{0\}}$ coincides with the global attractor of G .

Now let us consider problem (2.1) under conditions (2.2) and, additionally, let us assume that

$$\|d\|_+^2 := \sup_{t \geq 0} \int_t^{t+1} \|d(s)\|^2 ds < \infty. \quad (4.4)$$

It is known [4] that (4.4) implies that the set

$$\Sigma(d) = cl_{L_{loc}^{2,w}(\mathbb{R}_+; L^2(\Omega))} \{d(\cdot + h) \mid h \geq 0\}$$

is shift-invariant and compact in $L_{loc}^{2,w}(\mathbb{R}_+; L^2(\Omega))$. It means that every sequence $\{\sigma_n\} \subset \Sigma(d)$ has a weak convergence subsequence in $L_{loc}^2(\mathbb{R}_+; L^2(\Omega))$.

Moreover,

$$\forall \sigma \in \Sigma(d) \quad \|\sigma\|_+ \leq \|d\|_+.$$

Considering (2.1) with disturbance function $\sigma \in \Sigma(d)$ (we denote it by $(2.1)_\sigma$), we obtain by a simple calculation that the family $\{U_\sigma\}_{\sigma \in \Sigma}$, given by (3.8), is the m-semiprocesses family, i.e., satisfies Definition 4.1.

Theorem 4.1. *Under conditions (2.2), (4.4), the m-semiprocesses family of sets $\{U_\sigma\}_{\sigma \in \Sigma(d)}$ given by (3.8) possesses uniform attractor $\Theta_{\Sigma(d)}$, which satisfies invariance property (4.3).*

Proof. From inequality (3.2) and assumption (4.4) we deduce that $\forall u_0 \in L^2(\Omega)$, $\forall \sigma \in \Sigma(d)$, for every solution of $(2.1)_\sigma$ with $u|_{t=0} = u_0$, and $\forall t \geq 0$

$$\begin{aligned} \|u(t)\|^2 &\leq \|u_0\|^2 e^{-\delta t} + \frac{C}{\delta} + C \int_0^t e^{\delta(s-t)} \|d(s)\|^2 ds \\ &\leq \|u_0\|^2 e^{-\delta t} + \frac{C}{\delta} + C \|d\|_+^2 \cdot \left(1 + \frac{1}{\delta}\right). \end{aligned} \quad (4.5)$$

Estimate (4.5) implies dissipativity condition 1) of Lemma 4.1 with the set

$$B_0 = \left\{ u \in X \mid \|u\|^2 \leq 1 + \frac{C}{\delta} + C \|d\|_+^2 \left(1 + \frac{1}{\delta}\right) \right\}. \quad (4.6)$$

Considering condition 2) of Lemma 4.1, we have that for sufficiently large n

$$\begin{aligned} \xi_n \in U_{\sigma_n}(t_n, 0, u_n) &= U_{\sigma_n}(t_n - 1 + 1, t_n - 1, U_{\sigma_n}(t_n - 1, 0, u_n)) \\ &\subset U_{\sigma_n(\cdot + t_n - 1)}(1, 0, B_0). \end{aligned}$$

So, for some $\bar{\sigma}_n \in \Sigma(d)$, $\bar{u}_n \in B_0$ we can assume that $\xi_n = u_n(1)$, where u_n is a solution of $(2.1)_{\sigma_n}$, $u_n|_{t=0} = \bar{u}_n$.

Then inequalities $\|\bar{\sigma}_n\|_+ \leq \|d\|_+$ and (3.2) give that the sequence of solutions $\{u_n\}$ and the corresponding functions $\{l_n\}$, $l_n(t, x) \in f(u_n(t, x))$ satisfy (3.3)–(3.6) on the time interval $(0, T)$, $T > 1$.

In particular,

$$\bar{\sigma}_n \rightarrow \bar{\sigma} \text{ weakly in } L^2(Q_T),$$

$$\begin{aligned}
l_n &\rightarrow l \text{ weakly in } L^q(Q_T), \\
u_n &\rightarrow u \text{ in } L^2(Q_T), \\
u_n(t) &\rightarrow u(t) \text{ in } L^2(\Omega) \text{ for a.a. } t \in (0, T), \\
u_n(t) &\rightarrow u(t) \text{ weakly in } L^2(\Omega) \quad \forall t \in [0, T],
\end{aligned} \tag{4.7}$$

where u is a solution of (2.1) $_{\bar{\sigma}}$, $l(t, x) \in f(u(t, x))$ a.e. on Q_T .

As a direct consequence we obtain that for a.e. $t \in (0, T)$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u_n(t)\|^2 &\leq C|\Omega| + (\bar{\sigma}_n(t), u_n(t)), \\
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 &\leq C|\Omega| + (\bar{\sigma}_n(t), u(t)).
\end{aligned} \tag{4.8}$$

These inequalities and (4.7) mean that continuous functions

$$\begin{aligned}
J_n(t) &:= \|u_n(t)\|^2 - 2C|\Omega|t - 2 \int_0^t (\bar{\sigma}_n(s), u_n(s)) ds, \\
J(t) &:= \|u(t)\|^2 - 2C|\Omega|t - 2 \int_0^t (\bar{\sigma}(s), u(s)) ds
\end{aligned}$$

are monotonically decreasing, and convergent almost everywhere. Then Dini's Lemma guarantees pointwise convergence, in particular

$$J_n(1) \rightarrow J(1) \text{ as } n \rightarrow \infty. \tag{4.9}$$

It means that

$$\|u_n(1)\| \rightarrow \|u(1)\| \text{ as } n \rightarrow \infty. \tag{4.10}$$

As $u_n(1) \rightarrow u(1)$ weakly in $L^2(\Omega)$, so

$$\xi_n = u_n(1) \rightarrow u(1) \text{ in } L^2(\Omega) \tag{4.11}$$

and condition 2) of Lemma 4.1 is proved.

Let us verify 3). Assume that for fixed $\bar{t} > 0$

$$\xi_n \in U_{\sigma_n}(\bar{t}, 0, u_n^0), \text{ where } \sigma_n \rightarrow \sigma \text{ in } \Sigma(d), u_n^0 \rightarrow u^0 \text{ in } X$$

$$\text{and } \xi_n \rightarrow \xi \text{ in } X.$$

Then $\xi_n = u_n(\bar{t})$, where $u_n(\cdot)$ is a solution of (2.1) $_{\sigma_n}$, $u_n|_{t=0} = u_n^0$.

Repeating for the sequence $\{u_n\}$ arguments (4.7)–(4.11) on the interval $(0, T)$ $T > \bar{t}$ we obtain that up to subsequence

$$\xi_n = u_n(\bar{t}) \rightarrow \xi = u(\bar{t}) \in U_{\sigma}(\bar{t}, 0, u^0).$$

The theorem is proved. \square

Corollary 4.1.

$$\Theta_{\Sigma(d)} \subset B_0. \quad (4.12)$$

5. Asymptotic gain property for (2.1)

In this section, for bounded disturbances we prove AG stability of the global attractor Θ w.r.t. disturbances.

Assume that $d \in L^\infty(0, +\infty; L^2(\Omega))$, and

$$\|d\|_\infty := \operatorname{ess\,sup}_{t \geq 0} \|d(t)\| \leq R, \quad (5.1)$$

where $R \geq 0$ is a fixed number.

Theorem 5.1. *Under conditions (2.2), (5.1) there exists $\gamma \in \mathcal{K}$ such that $\forall u_0 \in X$*

$$\overline{\lim}_{t \rightarrow \infty} \|U_d(t, 0, u_0)\|_\Theta \leq \gamma(\|d\|_\infty), \quad (5.2)$$

where Θ is the global attractor of the unperturbed problem (2.1).

Proof. Condition (5.1) implies (4.4) with $\|d\|_+ \leq \|d\|_\infty$. So, from Theorem 4.1 we deduce that the m-semiprocesses family $\{U_\sigma\}_{\sigma \in \Sigma(d)}$ possesses uniform attractor $\Theta_{\Sigma(d)}$, which satisfies invariance property (4.3), and $\Theta_0 = \Theta$.

Let us prove that

$$\operatorname{dist}(\Theta_{\Sigma(d)}, \Theta) \rightarrow 0 \text{ as } \|d\|_\infty \rightarrow 0. \quad (5.3)$$

Assume the contrary. Then for some $\|d_n\|_\infty \rightarrow 0$ and $\xi_n \in \Theta_{\Sigma(d_n)}$

$$\operatorname{dist}(\xi_n, \Theta) \geq \varepsilon > 0. \quad (5.4)$$

Due to (4.3) we have that for every $T > 0$

$$\xi_n \in U_{\sigma_n}(T, 0, \eta_n),$$

where $\sigma_n \in \Sigma(d_n)$, $\eta_n \in \Theta_{\Sigma(d_n)}$.

From the inequality

$$\|\sigma_n\|_+ \leq \|d_n\|_+ \leq \|d_n\|_\infty$$

we deduce that

$$\sigma_n \rightarrow 0 \text{ in } L^2(Q_T) \text{ as } n \rightarrow \infty.$$

From (4.12), (5.1) we deduce that

$$\|\eta_n\|^2 \leq 1 + \frac{C}{\delta} + C \left(1 + \frac{1}{\delta}\right) \|d_n\|_+^2 \leq 1 + \frac{C}{\delta} + C \left(1 + \frac{1}{\delta}\right) R^2.$$

So, $\xi_n = u_n(T)$, where $u_n(\cdot)$ is a solution of (2.1) $_{\sigma_n}$.

Then, repeating arguments (4.7)–(4.11) on the interval $(0, T)$, we get that up to subsequence

$$\xi_n = u_n(T) \rightarrow \xi = u(T) \in U_0(T, 0, B_R) = G(T, B_R), \quad (5.5)$$

where

$$B_R = \{\eta \in X \mid \|\eta\|^2 \leq 1 + \frac{C}{\delta} + C \left(1 + \frac{1}{\delta}\right) R^2\},$$

G is the m-semiflow generated by unperturbed problem (2.1) (see Remark 4.1).

Using the attracting property

$$\text{dist}(G(t, B_R), \Theta) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

we can choose $T > 0$ such that

$$\text{dist}(G(T, B_R), \Theta) < \frac{\varepsilon}{2}. \quad (5.6)$$

So, $\xi_n \rightarrow \xi$ and $\text{dist}(\xi, \Theta) < \frac{\varepsilon}{2}$, which is a contradiction with (5.4).

We denote

$$\gamma_0(s) := \sup_{\|d\|_\infty \leq s} \text{dist}(\Theta_{\Sigma(d)}, \Theta).$$

Due to (4.12) $\Theta_{\Sigma(d)} \subset B_s$. So, the function $s \rightarrow \gamma_0(s)$ is correctly defined on $[0, +\infty)$, is non decreasing, and due to (5.3) $\gamma_0(s) \rightarrow 0$ $s \rightarrow 0$.

Then the classical comparison result [21] implies the existence of $\gamma \in \mathcal{K}$ such that $\gamma_0(s) \leq \gamma(s)$.

Since $d \in \Sigma(d)$, so for every $u_0 \in X$ we have

$$\begin{aligned} \|U_d(t, 0, u_0)\|_\Theta &= \text{dist}(U_d(t, 0, u_0), \Theta) \\ &\leq \text{dist}(U_d(t, 0, u_0), \Theta_{\Sigma(d)}) + \text{dist}(\Theta_{\Sigma(d)}, \Theta) \\ &\leq \text{dist}(U_d(t, 0, u_0), \Theta_{\Sigma(d)}) + \gamma(\|d\|_\infty). \end{aligned}$$

Passing to the upper limit and using attraction property (4.1), we obtain (5.2). Theorem is proved. \square

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