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ON UNIQUENESS AND CONTINUOUS DEPENDENCE OF THE WEAK SOLUTIONS OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONAL SPACES

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Abstract. In this paper we establish uniqueness of the weak solutions of nonlinear stochastic functional differential equations of neutral type in Hilbert spaces. We also study the continuous dependence of such solutions on the initial data.

Key words: functional differential equation, neutral type, Wiener process, weak solution, mild solution, weak convergence..

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1. Introduction

We study weak solutions of neutral type for stochastic functional differential equations of the form

$$d(u(t) - g(u_t)) = (Au + f(u_t))\,dt + \sigma(u_t)\,dW(t),\ t \in [0, T], \tag{1.1}$$

$$u(t) = \phi(t),\ t \in [-h, 0]. \tag{1.2}$$

Here A is a linear (unbounded) operator on a separable Hilbert space H, the noise $W(t)$ is a Q-Wiener process on a separable Hilbert space K.

For any $h > 0$ we denote by $C := C([-h, 0], H)$ to be the space of continuous H-valued functions $\phi : [-h, 0] \mapsto H$ with the norm

$$\|\phi\|_C := \sup_{\theta\in[-h,0]} \|\phi(\theta)\|_H,$$

where $\|\cdot\|_H$ stan ds for the norm in H. Throughout this paper we will be dropping the subscript H and denote $\|\cdot\|_H$ with $\|\cdot\|$. The solution of (1.1) is sometimes

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referred as a *state process.* We also denote $u_t := u(\theta + t)$, $\theta \in [-h, 0]$. Then $g : C \mapsto H$, $f : C \mapsto H$, and $\sigma : C \mapsto L_2^0$, where $L_2^0 = L(Q^{\frac{1}{2}}K, H)$ is the space of Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ to H. Finally, $(\Omega, \mathcal{F}, P)$ is a complete probability space, and $\phi : [-h, 0] \times \Omega \mapsto H$ is a given initial function.

Stochastic functional differential equations of neutral type usually describe mathematical models of different complex processes, whose evolution is affected by the previous states and is a subject of the influence of random forces. One example of such model is the Gurtin and Pipkin model [10], linearized version of which is

$$\frac{\partial}{\partial t}u(t,x) + \int_0^{+\infty} \beta(s)\frac{\partial}{\partial t}u(t-s,x)\,ds = a\Delta u(t,x).$$

In contrast with the classic heat equation, the delay term here describes the thermal history of the material and leads to the finite speed of heat propagation. This equation is a standard example of a functional-differential equation of neutral type. Similar memory effects emerge in Hodkin-Huxley model, Dawson-Fleming model of population genetics [6], modeling of population dynamics with delay of a birth process [23] among others.

The natural first question in the analysis of (1.1) is to establish its well-posedness as well as the continuous dependence on the initial conditions. In [16] the authors established the existence of mild solutions under the Lipschitz conditions and the linear growth conditions on nonlinearities. These conditions were somewhat relaxed in [12]. The sufficient conditions for the existence of mild solution were established in [24]. However, the aforementioned conditions are not easy to verify, since they are expressed through fractional degrees of unbounded operators. In [27] the authors obtained the existence results for stochastic reaction-diffusion equations of neutral type under less restrictive conditions. The work [29] established the comparison theorems for such equations. However, in all papers mentioned in this paragraph, the delay was of a special type - namely, as a function $\rho(t) \leq t$.

For general delays, in [28] the authors derived the conditions on existence and uniqueness of mild solutions, using the same approach as in [24]. Furthermore, in this work the authors investigated the asymptotic behavior of solutions in the invariant measure sense. The problem of asymptotic behavior of stochastic functional-differential equations with elliptic operator in both bounded and unbounded domains was considered in [4] and for finite-dimensional case in [30], [18].

The weak well-posedness question for stochastic functional differential equations of non-neutral type (i.e., when in equation (1.1) $g = 0$), was addressed in e.g. [2], [3]. In these works, the nonlinear terms f and σ are required to be Lipschitz, which is a fairly restrictive assumption. In the paper [22] the Lipschitz condition was relaxed and local monotonicity condition was used instead. In [27] the authors, using the approach from [22], establish the existence and uniqueness of weak and strong solutions for a system of stochastic functional differential equations.

To the best of our knowledge, the question of existence and uniqueness of weak solutions for equation (1.1) has not been addressed in the literature. Hence, the main result of this article is deriving the uniqueness conditions for its solution, and its continuous dependence on the initial data. One of the difficulties arising in the process of establishing these results, is the fact that the nonlinear term f is not Lipschitz continuous, but only satisfies the monotonicity condition. Furthermore, in [22] the authors use the Ito formula for $\|u(t)\|$, while here we need to use the Ito formula for $\|u(t) - g(u_t)\|$. This creates additional non-monotonic terms of the form $(f(u_t), g(u_t))$, which ought to be estimated.

The paper is structured as follows. In Section 2 below we introduce the notations, some auxiliary results, and formulate the main results. The proofs of the main results are in Section 3.

2. Preliminaries

Let K, H, V be real separable Hilbert spaces,

$$V \subset H \subset V^{'}$$

be a Gelfand triple, where the inclusion $V \subset H$ is dense and compact, and $V^{'}$ is the dual space to V. The norm in V we denote by $\|\cdot\|_V$, the inner product in H we denote as $(\cdot,\cdot)$, and $\langle \cdot,\cdot \rangle$ denotes V - $V^{'}$ pairing.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t; t \geq 0\}$ generated by the Q-Wiener process W on $(\Omega, \mathcal{F}, P)$ with linear bounded covariance operator such that $trQ < \infty$.

We assume that there exist a complete ortonormal system $\{e_k\}$ in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...,$ and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k,$$

where $\{\beta_k(t)\}$ are real-valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$.

Let $U_0 = Q^{\frac{1}{2}}(K)$ and $L_2^0 = L_2(U_0, H)$ be the space of all Hilbert-Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi)_{L_2^0} = tr[\Phi Q \Psi^*]$ and the norm $\|\phi\|_{L_2^0}$ respectively.

Assume that $A : V \mapsto V^{'}$ is a self-adjoint and coercive linear operator and $a(u, v) = \langle -Au, v\rangle$, $u, v \in V$ is the corresponding bilinear form. We define the domain of A in H as $D(A) := \{u \in V; Au \in H\}$. This allows us to consider A as a linear unbounded operator in H with domain $D(A)$. Then $(-A)^{-1}$ can be considered as a self-adjoint compact operator in H. Consequently, there exists a complete orthonormal basis in H $\{v_i\} \subset D(A)$, and a sequence $\zeta_i \longrightarrow \infty$, $i \longrightarrow \infty$ such that

$$0 < \zeta_1 \leq \zeta_2 \leq ...,$$

and

$$(-A)v_i = \zeta_i v_i.$$

This way the scalar product and the norm in V become

$$(u, v)_V = \sum_{j=1}^{\infty} \zeta_j (u, v_j) \cdot (v, v_j),$$

and

$$\|u\|_V^2 = \sum_{j=1}^{\infty} \zeta_j (u, v_j)^2.$$

respectively. We note that the system $\{v_i\}$ is complete in V, with $(v_i, v_k)_V = \delta_{j,k}$ being the Kronecker symbol.

We assume that A is infinitesimal generator of an analytic semigroup $S(t) = e^{At}$ of bounded in H operators. Note that this is the case if and only if $(-A)$ is a sectorial operator [11].

Taking into account that $(-A)^{-1}$ is a compact operator, we deduce that $S(t)$ is a compact semigroup ([11], Section 1.4). Hence, Theorem 3.2 in [19] implies that $S(t)$ is continuous in the uniform operator topology. From [19] we can deduce that for all $\alpha \in (0, 1)$ the fractional power of the operator $(-A)$ is a closed linear operator with domain $D(-A^{\alpha})$.

In the work we make use of the following proposition, which is a direct consequence of [11], Theorem 1.4.3.

Proposition 2.1. There exist $C_\alpha > 0$ such that

$$\|(-A)^{\alpha} S(t)\| \leq C_\alpha t^{-\alpha} e^{-\zeta_1 t},$$

for all $t > 0$.

Let $L_2^V := L^2([-h, 0]; V)$ be the space of V-valued mappings with the norm

$$\|u\|_{L_2^V}^2 := \int_{-h}^{0} \|u(t)\|_V^2 \, dt.$$

Lemma 2.1. *[28] Let $u(t), -h \leq t \leq T$ be a stochastic process with continuous trajectories. Then, if $\alpha \geq 1$ and $u_0 = \phi$, we have*

$$\sup_{t \in [0,T]} \|u_t\|_C^{\alpha} \leq \|\phi\|_C^{\alpha} + \sup_{t \in [0,T]} \|u(t)\|^{\alpha}.$$

To ensure the existence and uniqueness of solution, we have to impose additional conditions on the operator A and the mappings f, σ, g.
Conditions on A:

1. $A : V \mapsto V'$ is a linear bounded self-adjoint operator.
2. There exists $\alpha > 0$ such that for any $u, v \in V$ we have

$$|\langle Au, v\rangle| \leq \alpha \|u\|_V \|v\|_V.$$

3. A satisfies the coercitivity condition: there exists $\beta > 0$ such that

$$\langle Av, v\rangle \leq -\beta \|v\|_V^2, \ \forall v \in V.$$

These conditions imply that A is infinitesimal generator of an analytic semigroup $S(t) = e^{At}$ of bounded in H operators.

Conditions on g:

1. g is a mapping from $C \cap L_2^V$ to V.
2. (Growth condition)
   (a) There exists $K_1 > 0$ such that for all $\phi \in C \cap L_2^V$ we have

$$\|g(\phi)\|_V \leq K_1(1 + \|\phi\|_{L_2^V} + \|\phi\|_C);$$

   (b) There exist $K_2 > 0, \lambda \in (0, 1)$ such that for all $\phi \in C \cap L_2^V$ we have

$$\|g(\phi)\|^2 \leq K_2 \left(1 + \int_{-h}^{0} \|\phi(\theta)\|^2 \, d\theta\right) + \lambda \|\phi(0)\|^2.$$

3. There exist $L \in (0, \frac{1}{4(h+1)})$ such that for all $\phi \in C \cap L_2^V$ we have

$$\|g(\phi) - g(\psi)\|^2 \leq L \left(\int_{-h}^{0} \|\phi(\theta) - \psi(\theta)\|^2 \, d\theta + \|\phi(0) - \psi(0)\|^2\right).$$

   As a direct consequence, we obtain that there exist $L_1 \in (0, \frac{1}{4})$ such that for all $\phi \in C \cap L_2^V$

$$\|g(\phi) - g(\psi)\|^2 \leq L_1 \|\phi - \psi\|_C^2.$$

4. There exists $L_2 \in (0, \frac{\beta}{\alpha})$ such that for all $\phi \in C \cap L_2^V$ we have

$$\|g(\phi) - g(\psi)\|_V \leq L_2 \left(\int_{-h}^{0} \|\phi(\theta) - \psi(\theta)\|_V \, d\theta + \|\phi(0) - \psi(0)\|_V\right) + (\rho(\phi) + \rho(\psi))\|\phi - \psi\|_C,$$

   where $\rho : C \cap L_2^V \mapsto [0, +\infty)$ and exist $K_3 > 0, \gamma \geq 1$ such that

$$\rho(\phi) \leq K_3 \left(\|\phi\|_C^\gamma + \left(\int_{-h}^{0} \|\phi(\theta)\|_V \, d\theta\right)^\gamma\right).$$

Conditions on f and σ:

1. f is a mapping from $C \cap L_2^V$ to H, σ is a mapping from $C \cap L_2^V$ to L_2^0.
2. (Growth condition) There exist $K_4, K_5 > 0, \gamma \geq 1$ such that we have
$$\|f(\phi)\| \leq K_4 \left(1 + \left(\int_{-h}^0 \|\phi(t)\|_V \, dt\right)^\gamma + \|\phi\|_C^\gamma + \|\phi(0)\|_V^\gamma\right),$$
$$\|\sigma(\phi)\|_{L_2^0}^2 \leq K_5(1 + \|\phi\|_C^2),$$
for all $\phi \in C \cap L_2^V$.
3. There exists $\nu > 0$ such that for any $\phi \in C \cap L_2^V$ we have
$$(f(\phi), \phi(0)) \leq \nu(\|\phi\|_C^2 + 1).$$
4. (Monotonicity condition) There exists $\delta > 0$ such that for any $\phi, \psi \in C \cap L_2^V$ we have
$$\|\sigma(\phi) - \sigma(\psi)\|_{L_0^2}^2 \leq \delta \|\phi - \psi\|_C^2,$$
$$(f(\phi) - f(\psi), \phi(0) - \psi(0)) \leq (\rho(\phi) + \rho(\psi))\|\phi - \psi\|_C^2,$$
$$\|f(\phi) - f(\psi)\| \leq (\rho(\phi) + \rho(\psi)) \left(\|\phi(0) - \psi(0)\|_V + \int_{-h}^0 \|\phi(\theta) - \psi(\theta)\|_V \, d\theta\right),$$
where ρ is given in (G4).
5. There exists $K_6 > 0$ such that the following inequality holds
$$(f(\phi), g(\phi)) \leq K_6(1 + \|\phi\|_C^2).$$

We denote $\Omega_T := \Omega \times [0, T]$.

Definition 2.1. For a given $\mathcal{F}_0$-measurable random variable $\phi \in C$ an $\mathcal{F}_t$ adapted random process $u(t) \in V$ is a weak solution of (1.1),(1.2) on $[0, T]$ if:

1. $u(t) = \phi(t), t \in [-h, 0]$,
2. $u \in L^2(\Omega_T; V) \cap L^2(\Omega; C([0, T]; H))$,
3. for any $v \in V$ we have
$$(u(t) - g(u_t), v) = (\phi(0) - g(\phi), v) + \int_0^t \langle Au(s), v \rangle \, ds + \int_0^t (f(u_s), v) \, ds + \int_0^t (\sigma(u_s) \, dW(s), v),$$
for each $t \in [0, T]$ a.s.

The main results will be proved under the assumption that the following proposition holds:

Proposition 2.2. (Existence)
Suppose that conditions (A1)-(A3), (G1)-(G4) and (N1)-(N5) hold. Then for any $\mathcal{F}_0$ measurable initial condition $\phi \in C \cap L_2^V$ which satisfies condition

$$\mathbf{E}\left(\|\phi\|_C^{2\gamma} + \left(\int_{-h}^{0} \|\phi(\theta)\|_V^2 d\theta\right)^{\gamma}\right) < \infty,$$

the problem (1.1)),(1.2) has a weak solution on $[0,T]$ such that for some constant dependant on $\phi \in C$

$$\mathbf{E}\left(\sup_{t\in[0,T]} \|u(t)\|^{2\gamma} + \left(\int_{0}^{T} \|u(t)\|_V^2 dt\right)^{\gamma}\right) < C(\phi). \tag{2.1}$$

Moreover, the energy equality holds:

$$\begin{aligned}\|u(t) - g(u_t)\|^2 = \|\phi(0) - g(\phi)\|^2 \\ + \int_0^t \big\langle Au(s), u(s) - g(u_s)\big\rangle\, ds + \int_0^t \Big(f(u_s), u(s) - g(u_s)\Big)\, ds \\ + \int_0^t \|\sigma(u_s)\|_{L_2^0}^2\, ds + \int_0^t \Big(\sigma(u_s)\, dW(s), u(s) - g(u_s)\Big).\end{aligned}$$

This proposition will be proven in detail in forthcoming article. Its proof can be structured as follows:
Step 1. Prove the solvability of the finite-dimensional problem for the Galerkin projections. To this end, we use the Euler approximation sequence, which allows to prove existence of a martingal solution. Next, considering trajectory-wise uniqueness of solution and Jamada-Watanabe theorem we prove the existence of weak solution for the Galerkin projected equation.
Step 2. Obtain uniform a priori estimates for the Galerkin projected solutions.
Step 3. Prove the existence of solution of (1.1) by passing to the limit in the Galerkin approximations.

The main results of this work is the following two theorems.

Theorem 2.1. *(Uniqueness)*
Under conditions from Proposition 2.2, the solution is unique

Theorem 2.2. *(Continious dependence on initial data)*
Assume that the conditions of Proposition 2.2 hold. Let ϕ^n, ϕ be initial conditions in (1.2)*, such that*

$$\lim_{n\to\infty} \mathbf{E}\left(\|\phi^n - \phi\|_C^2 + \int_{-h}^{0} \|\phi^n(t) - \phi(t)\|_V^2\, dt\right) = 0.$$

Let $u^n(t), u(t)$ be the corresponding solutions of (1.1)*),*(1.2)*. Then the following property holds:*

$$\lim_{n\to\infty} \mathbf{E} \sup_{t\in[0,T]} \|u^n(t) - u(t)\|^2 = 0.$$

3. Proof of the main results

3.1. Uniqueness

Proof of Theorem 2.1. Suppose, that $u(t)$ and $v(t)$ are two different solutions of the initial value problem (1.1),(1.2) on $[0,T]$ with the same initial function ϕ. Then, using the definition of solution and the Ito formula, we have

$$
\begin{aligned}
&\|(u(t) - g(u_t)) - (v(t) - g(v_t))|^2 \\
&= 2\int_0^t \Big((u(s) - g(u_s)) - (v(s) - g(v_s)), (\sigma(u_s) - \sigma(v_s))\, dW(s)\Big) \\
&\quad + 2\int_0^t \big\langle A(u(s)) - A(v(s)), (u(s) - g(u_s)) - (v(s) - g(v_s))\big\rangle\, ds \\
&\quad + 2\int_0^t \Big(f(u_s) - f(v_s), (u(s) - g(u_s)) - (v(s) - g(v_s))\Big)\, ds \\
&\quad + \int_0^t \|\sigma(u_s) - \sigma(v_s)\|^2\, ds.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\|(u(t) - g(u_t)) - (v(t) - g(v_t))\|^2 \\
&= \|u(t) - v(t)\|^2 + \|g(u_t) - g(v_t)\|^2 - 2(u(t) - v(t), g(v_t) - g(u_t)).
\end{aligned}
$$

This way,

$$
\begin{aligned}
&\|u(t) - v(t)\|^2 \\
&\le 2\int_0^t \Big((u(s) - g(u_s)) - (v(s) - g(v_s)), \sigma(u_s) - \sigma(v_s)\Big)\, dW(s) \\
&\quad + 2\int_0^t \big\langle A(u(s)) - A(v(s)), u(s) - (v(s)\big\rangle\, ds \\
&\quad + 2\int_0^t \big\langle A(u(s)) - A(v(s)), g(v_s) - g(u_s)\big\rangle\, ds \\
&\quad + 2\int_0^t (f(u_s) - f(v_s), u(s) - v(s))\, ds + 2\int_0^t (f(u_s) - f(v_s), g(v_s) - g(u_s))\, ds \\
&\quad + \int_0^t \|\sigma(u_s) - \sigma(v_s)\|^2\, ds + 2(u(t) - v(t), g(v_t) - g(u_t)).
\end{aligned}
$$

For any $R > 0$ define the stopping time

$$
\tau_R = \inf\left\{t \in [0,T] :\ \int_0^t (\rho^2(u_s) + \rho^2(v_s))\, ds > R\right\} \wedge T, \tag{3.1}
$$

where $a \wedge b = \min\{a, b\}$.

Since both u and v solve (1.1),(1.2) on $[0,T]$, we deduce that

$$\lim_{R\to\infty}\tau_R = T,\ a.s.$$

For any $\tau\in[0,T]$, we have

$$\begin{aligned}
&\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2\\
&\le \mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]} 2\int_0^t \Big((u(s)-g(u_s))-(v(s)-g(v_s)),\sigma(u_s)-\sigma(v_s)\Big)\,dW(s)\\
&\quad+\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]} 2(u(t)-v(t),g(v_t)-g(u_t))\\
&\quad+2\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t -\beta\|u(s)-v(s)\|_V^2\,ds\\
&\quad+2\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t \alpha\|u(s)-v(s)\|_V\cdot\|g(u_s)-g(v_s)\|_V\,ds\\
&\quad+2\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t (\rho(u_s)+\rho(v_s))\|u_s-v_s\|_C^2\,ds\\
&\quad+2\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t \Big(f(u_s)-f(v_s),g(v_s)-g(u_s)\Big)\,ds\\
&\quad+\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t \|\sigma(u_s)-\sigma(v_s)\|_{L_0^2}^2\,ds.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&2\sup_{t\in[0,\tau\wedge\tau_R]}|(u(t)-v(t),g(v_t)-g(u_t))|\\
&\le \frac{1}{2}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2+2L_1\sup_{t\in[0,\tau\wedge\tau_R]}\|u_t-v_t\|_C^2.
\end{aligned}$$

To estimate the stochastic integral, we use the Buckholder-Devis-Gundy inequality ([20] Theorem 5.2.4), Lemma 2.4.2 [20], and Lemma 2.1.

$$\begin{aligned}
&\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]} 2\Big|\int_0^t \Big((u(s)-g(u_s))-(v(s)-g(v_s)),\sigma(u_s)-\sigma(v_s)\Big)\,dW(s))\Big|\\
&\le C\mathbf{E}\left(\int_0^{\tau\wedge\tau_R}\Big(\big(\|u(s)-v(s)\|+\|g(u_s)-g(v_s)\|\big)^2\cdot\|\sigma(u_s)-\sigma(v_s)\|_{L_0^2}^2\Big)^{\frac{1}{2}}\,ds\right)\\
&\le C\mathbf{E}\left(\int_0^{\tau\wedge\tau_R}\Big(\big(\|u(s)-v(s)\|+L_1^{\frac{1}{2}}\|u_s-v_s\|_C\big)^2\cdot\delta\|u_s-v_s\|_C^2\Big)^{\frac{1}{2}}\,ds\right)\\
&\le C\mathbf{E}\left(\int_0^{\tau\wedge\tau_R}\Big(\sup_{t\in[0,s]}\|u(t)-v(t)\|^4\Big)^{\frac{1}{2}}\,ds\right)\\
&\le \varepsilon C\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2+\frac{C}{\varepsilon}\mathbf{E}\int_0^{\tau\wedge\tau_R}\sup_{t\in[0,s]}\|u(t)-v(t)\|^2\,ds,
\end{aligned}$$

for some positive constant $C > 0$ and $\varepsilon > 0$. Next, we get

$$\begin{aligned}
&\alpha\|u(s)-v(s)\|_V \cdot \|g(u_s)-g(v_s)\|_V \\
&\le \alpha\|u(s)-v(s)\|_V\Big(L_2\left(\int_{-h}^0 \|u(s+\theta)-v(s+\theta)\|_V d\theta + \|u(s)-v(s)\|_V\right) \\
&\quad + (\rho(u_s)+\rho(v_s))\|u_s-v_s\|_C\Big) \\
&\le \alpha L_2\|u(s)-v(s)\|_V^2 + \varepsilon\alpha\|u(s)-v(s)\|_V^2 \\
&\quad + \frac{L_2}{\varepsilon}\int_{-h}^0 \|u(s+\theta)-v(s+\theta)\|_V^2 d\theta + \frac{\rho^2(u_s)+\rho^2(v_s)}{\varepsilon}\|u_s-v_s\|_C^2.
\end{aligned}$$

We remark that

$$\int_0^t\int_{-h}^0 \|u(s+\theta)-v(s+\theta)\|_V^2 d\theta\, ds \le t\int_0^t \|u(s)-v(s)\|_V^2\, ds.$$

From (N4) for all $t \in [0, \tau\wedge\tau_R]$ we have

$$\begin{aligned}
&2\int_0^t |(f(u_s)-f(v_s), g(v_s)-g(u_s)| \le 2\int_0^t \|f(u_s)-f(v_s)\|\cdot\|g(v_s)-g(u_s)\|\, ds \\
&\le 2\int_0^t \Big((\rho(u_s)+\rho(v_s))(\|u(s)-v(s)\|_V \\
&\quad + \int_{-h}^0 \|u(s+\theta)-v(s+\theta)\|_V d\theta)\Big) L_1^{\frac{1}{2}}\|u_s-v_s\|_C\, ds \\
&\le 2\int_0^t \Big(\varepsilon\|u(s)-v(s)\|_V^2 + \frac{L_1}{\varepsilon}(\rho^2(u_s)+\rho^2(v_s))\|u_s-v_s\|_C^2 \\
&\quad + L_1(\rho^2(u_s)+\rho^2(v_s))\|u_s-v_s\|_C^2 + \frac{h}{2}\int_{-h}^0 \|u(s+\theta)-v(s+\theta)\|_V^2 d\theta\Big)\, ds \\
&\le \int_0^t 2\varepsilon\|u(s)-v(s)\|_V^2\, ds + \int_0^t (\rho^2(u_s)+\rho^2(v_s))\left(\frac{2L_1}{\varepsilon}+2L_1\right)\|u_s-v_s\|_C^2\, ds \\
&\quad + ht\int_0^t \|u(s)-v(s)\|_V^2\, ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2 \\
&\le \frac{1}{2}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2 + \frac{1}{2}L_1\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2 \\
&\quad + \varepsilon C\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2 + \frac{C}{\varepsilon}\mathbf{E}\int_0^{\tau\wedge\tau_R}\sup_{t\in[0,s]}\|u(t)-v(t)\|^2\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\int_0^t\Big(-2\beta\|u(s)-v(s)\|_V^2\\
&+\left(4\alpha^2\varepsilon+2\alpha L_2+2\frac{L_2^2}{\varepsilon}t+2\varepsilon+2th\right)\|u(s)-v(s)\|_V^2\Big)\,ds\\
&+\mathbf{E}\sup_{t\in[\tau\wedge\tau_R]}\int_0^t\left(\delta+\rho(u_s)+\rho(v_s)+\left(\frac{4}{\varepsilon}+4\right)(\rho^2(u_s)+\rho^2(v_s))\right)\|u_s-v_s\|_C^2\,ds.
\end{aligned}
$$

Thus, using conditions on L_1 and L_2, for sufficiently small ε,τ we have

$$\frac{1}{2}+2L_1+\varepsilon C<1,$$

and

$$2\alpha^2\varepsilon+\alpha L_2+\frac{L_2^2}{\varepsilon}t+2\varepsilon+th<-\beta.$$

Thus, we get the inequality

$$
\begin{aligned}
&\mathbf{E}\sup_{t\in[0,\tau\wedge\tau_R]}\|u(t)-v(t)\|^2\\
&\le\mathbf{E}\int_0^{\tau\wedge\tau_R}\Big(\delta+\rho(u_s)+\rho(v_s)\\
&\quad+\frac{C}{\varepsilon}+\left(\frac{4}{\varepsilon}+4\right)(\rho^2(u_s)+\rho^2(v_s))\Big)\sup_{p\in[0,s]}\|u(p)-v(p)\|^2\,ds.
\end{aligned}
$$

Using the definition of τ_R, the expression in brackets is bounded. Thus, due to Lemma 2 [9], we conclude that for every $R>0$ $u(t)=v(t)$ for every $t\in[0,\tau_R\wedge\tau]$.

Passing to the limit as $R\to\infty$, we have that $u(t)=v(t)$ for $t\in[0,\tau]$.

By continuing this procedure on intervals $[\tau,2\tau]$, $[2\tau,3\tau]$ etc., we obtain the uniqueness on the interval $[0,T]$. □

3.2. Continuous dependence on initial data

We start with noting that using (N2) we have

$$
\begin{aligned}
&\mathbf{E}\int_0^{t_1}|f(u_s^n)\|^2\,ds\le\mathbf{E}\int_0^{t_1}K_4\left(1+\left(\int_{-h}^0\|u_s^n\|_V\,d\theta\right)^\gamma+\|u_s^n\|_C^\gamma+\|u^n(s)\|^\gamma\right)^2 ds\\
&\le\int_0^{t_1}2K_4\mathbf{E}\left(1+\left(\int_{-h}^0\|u_s^n\|_V^2\,d\theta\right)^\gamma+\sup_{s_1\in[0,s]}\|u^n(s_1)\|_C^{2\gamma}+\|\phi^n\|_C^{2\gamma}\right)^2 ds\\
&\le\int_0^{t_1}2K_4\left(1+\mathbf{E}\left(\int_{-h}^0\|u_s^n\|_V^2\,d\theta\right)^\gamma+\mathbf{E}\sup_{s_1\in[0,s]}\|u^n(s_1)\|^{2\gamma}+\mathbf{E}\|\phi^n\|_C^{2\gamma}\right)ds.
\end{aligned}
$$

The condition (2.1) yields that there exists $C>0$ such that

$$\mathbf{E}\int_0^{t_1}\|f(u_s^n)\|^2\,ds\le C. \tag{3.2}$$

Lemma 3.1. *Assume that the operator A satisfies conditions (A1)-(A3), $f(t)$ is a predictable process with integrable trajectories in H, $\sigma(t) \in L_2^0$, $t \in [0,T]$ is a predictable process, such that*

$$\mathbf{E}\int_0^T \|\sigma(s)\|_{L_2^0}^2\, ds < \infty,$$

and $g(t) \in V$, $t \in [0,T]$ is a predictable process with locally integrable trajectories in V. Then if $u(t)$ is a unique weak solution of the linear equation

$$d(u(t) - g(t)) = (Au + f(t))\, dt + \sigma(t)\, dW(t),\ t \in [0,T], \tag{3.3}$$

$$u(0) = u_0, \tag{3.4}$$

then we have

$$\begin{aligned} u(t) = S(t)(u(0) - g(0)) + g(t) + \int_0^t AS(t-s)g(s)\, ds \\ + \int_0^t S(t-s)f(s)\, ds + \int_0^t S(t-s)\sigma(s)\, dW(s). \end{aligned} \tag{3.5}$$

Proof of Lemma 3.1. The conditions of Lemma 3.1 ensure that all integrals in the right hand side of (3.5) are well-defined. Define $z(t) = u(t) - g(t)$. Since $u(t)$ is a weak solution, we have

$$\begin{aligned} (z(t), v) = (z(0), v) + \int_0^t \Big(\langle Az(s), v\rangle + \langle Ag(s), v\rangle + (f(s), v)\Big)\, ds \\ + \int_0^t \Big(\sigma(s)\, dW(s), v\Big), \forall v \in V. \end{aligned} \tag{3.6}$$

In (3.6) fix $v \in D(A)$ and consider the function $\zeta(t) = \phi(t)v$, $t \in [0,T]$, where $\phi(t)$ is a smooth real-valued function. Defining $\psi(t) = (z(t), v)$, from (3.6) we conclude that $\psi(t)$ has a stochastic differential on $[0,T]$. Therefore,

$$\psi(t) = \int_0^t \big(\langle Az(s), v\rangle + \langle Ag(s), v\rangle + (f(s), v)\big)\ ds + \int_0^t (\sigma(s)\, dW(s), v).$$

Applying the Ito's formula to the process $\psi(t)\phi(t)$, we get

$$\begin{aligned} (z(t), \zeta(t)) = (z(0), \phi(0)v) + \int_0^t \Big(\langle Az(s), \zeta(s)\rangle + \langle Ag(s), \zeta(s)\rangle \\ + (z(s), \zeta^{'}(s)) + (f(s), \zeta(s))\Big)\, ds + \int_0^t (\sigma(s)\, dW(s), \zeta(s)). \end{aligned} \tag{3.7}$$

Since the set of functions ζ is dense in $C^1([0,T],D(A))$, the formula (3.7) holds for arbitrary $\zeta \in C^1([0,T],D(A))$. Choosing $\zeta(s) = S(t-s)v$, $s \in [0,t]$, we have

$$(z(t),v) = (S(t)z(0),v) + \int_0^t \Big(\big\langle Az(s), S(t-s)v \big\rangle + \big\langle Ag(s), S(t-s)v \big\rangle - (z(s), AS(t-s)v) + (S(t-s)f(s),v) \Big)\, ds + \int_0^t (S(t-s)\sigma(s)\, dW(s), v). \tag{3.8}$$

Since $v \in D(A)$ and $S(t)$ is an analytic semigroup, we have

$$\big\langle Az(s), S(t-s)v \big\rangle = \big\langle z(s), AS(t-s)v \big\rangle = \big\langle z(s), S(t-s)Av \big\rangle = (z(s), S(t-s)Av),$$

and

$$\big\langle Ag(s), S(t-s)v \big\rangle = (g(s), S(t-s)Av) = (S(t-s)Ag(s), v).$$

Hence, (3.8) yields

$$(u(t)-g(t),v) = (S(0)(u(0)-g(0)),v) + \int_0^t \Big((AS(t-s)g(s),v) + (S(t-s)f(s),v) \Big)\, ds + \int_0^t (S(t-s)\sigma(s)\, dW(s), v).$$

Since the embedding $D(A) \subset H$ is dense, we have (3.5). This completes the proof of Lemma 2.2.

□

Proof of Theorem 2.2. Using Lemma 3.1, we have the following relations for $u^n(t)$ and $u(t)$:

$$u^n(t) = S(t)(\phi^n(0) - g(\phi^n)) + g(u_t^n) + \int_0^t AS(t-s)g(u_t^n)\, ds + \int_0^t S(t-s)f(u_s^n)\, ds + \int_0^t S(t-s)\sigma(u_s^n)\, dW(s), \tag{3.9}$$

and

$$u(t) = S(t)(\phi(0) - g(\phi)) + g(u_t) + \int_0^t AS(t-s)g(u_t)\, ds + \int_0^t S(t-s)f(u_s)\, ds + \int_0^t S(t-s)\sigma(u_s)\, dW(s).$$

From a priori estimates (2.1) there exists a subsequence (still denoted by $\{u^n\}$), and (u, Σ, F) such that

$$u^n \to u \text{ weakly } \text{ in } L^2(\Omega_T, V),$$

$$\sigma(u^n_s) \to \Sigma \text{ weakly } \text{ in } L^2(\Omega_T, L^0_2),$$

$$f(u^n_s) \to F \text{ weakly } \text{ in } L^2(\Omega_T, H).$$

Lemma 3.2. *[28](Lemma 3.6) The operator*

$$Bz := \int_0^t AS(t-s)z(s)\,ds$$

is compact from $C([0,T],V)$ into $C([0,T],H)$.

Lemma 3.3. *[20](Proposition 8.4) For any $p>1$ and $\beta \in (\frac{1}{p}, 1]$ the operator*

$$G_\beta z := \int_0^t (t-s)^{\beta-1} S(t-s)z(s)\,ds$$

is compact from $L^p((0,T),H)$ into $C([0,T],H)$.

For any function $f \in C([0,t_1],H)$, we denote:

$$V^{t_1}_\delta(f) := \sup_{t,s\in[0,t_1];|t-s|\le\delta} \|f(t)-f(s)\|.$$

Using (2.1) we deduce that for some $M>0$ and for all N we have

$$\begin{aligned}
&P\{\sup_{t\in[0,t_1]} \|u^n_t\|_{L^V_2} + \sup_{t\in[0,t_1]} \|u^n_t\|_C > N\} \le \frac{2}{N^2}\mathbf{E} \sup_{t\in[0,t_1]} \left(\|u^n_t\|^2_{L^V_2} + \|u^n_t\|^2_C\right)\\
&\le \frac{2}{N^2}\mathbf{E}\left(\sup_{t\in[0,t_1]} \int_{t-h}^t \|u^n_s\|^2_V\,ds + \sup_{t\in[0,t_1]} \|u^n_t\|^2\right)\\
&\le \frac{2}{N^2}\left(\int_{-h}^0 \mathbf{E}\|\phi^n(s)\|^2_V\,ds + \int_0^t \mathbf{E}\|u^n(s)\|^2_V\,ds + \mathbf{E}\|\phi\|^2_C + \mathbf{E}(\sup_{t\in[0,t_1]} \|u^n_t\|^2)\right)\\
&\le \frac{M}{N^2} \to 0, \text{ as } N\to\infty.
\end{aligned}$$

Thus, for any $\varepsilon>0$ there exists $N(\varepsilon)$ such that

$$P\{\sup_{t\in[0,t_1]} (\|u^n_t\|^2_{L^V_2} + \|u^n_t\|^2_C) > N(\varepsilon)\} \le \frac{\varepsilon}{2}.$$

For all $\varepsilon > 0$, we also have

$$\begin{aligned}
&P\{\sup_{t,s\in[0,t_1];|t-s|<\delta} \|g(u^n_t) - g(u^n_s)\| > \varepsilon\} \\
&\leq \frac{1}{\varepsilon^2}\mathbf{E} \sup_{t,s\in[0,t_1];|t-s|<\delta} \|g(u^n_t) - g(u^n_s)\|^2 \leq \frac{L_1}{\varepsilon^2}\mathbf{E}(\sup_{t,s\in[0,t_1];|t-s|<\delta} \|u^n_t - u^n_s\|^2_C) \\
&\leq \frac{L_1}{\varepsilon^2}\left(\mathbf{E}(\sup_{t,s\in[0,t_1];|t-s|<\delta} \|\phi^n_t - \phi^n_s\|^2_C) + \mathbf{E}(\sup_{t,s\in[0,t_1];|t-s|<\delta} \|u^n(t) - u^n(s)\|^2)\right) \\
&\to 0, \text{ as } \delta \to 0.
\end{aligned}$$

Then for arbitrary $\varepsilon > 0$ there exists $\delta_k \to 0$ such that

$$P\{\sup_{t,s\in[0,t_1];|t-s|<\delta_k} \|g(u^n_t) - g(u^n_s)\| > \varepsilon\} \leq \frac{\varepsilon}{2^{k+1}}.$$

Thus,

$$P\{\bigcap_{k=1}^{\infty}\{\omega : V^{t_1}_{\delta_k}(g(u^n_t)) \leq \frac{1}{k}\}\} > 1 - \frac{\varepsilon}{2}.$$

Denote by K_ε the subset of

$$A = \bigcap_{k=1}^{\infty}\{f : V^{t_1}_{\delta_k}(f)) \leq \frac{1}{k}\},$$

such that for all $t \in [0, t_1]$ the set $\{f(t); f \in A\}$ is compact in H, where is ε in this definition. Then K_ε is compact in $C([0, t_1], H)$. Moreover,

$$P\{g(u^n_t) \notin K_\varepsilon\}$$

$$\leq P\{\sup_{t\in[0,t_1]}(\|u^n_t\|_{L^V_2} + \|u^n_t\|^2_C) > N\} + P\{\bigcap_{k=1}^{\infty}\{V^{t_1}_{\delta_k}(g(u^n_t)) > \frac{1}{k}\}\} \leq \varepsilon.$$

Next, using the factorization formula [20](page 128) we have

$$\begin{aligned}
u^n(t) &= S(t)(\phi^n(0) - g(\phi^n)) + g(u^n_t) \\
&\quad + \int_0^t AS(t-s)g(u^n_s)\,ds + \left(G_1 f(u^n_s) + \frac{\sin(\beta\pi)}{\pi}(G_\beta Y_k(s))(t)\right),
\end{aligned}$$

where

$$Y_k(s) = \int_0^s (s-\tau)^{-\beta} S(s-\tau)\sigma(u^n_\tau)\,dW(\tau),\ \beta \in (0, 1).$$

But

$$\mathbf{E}\int_0^{t_1} \|Y(s)\|^p\,ds \leq C_0^p \mathbf{E}\int_0^{t_1}\left(\int_0^s (s-\tau)^{-2\beta}\|\sigma(u^n_\tau)\|^2_{L^0_2} d\tau\right)^{\frac{p}{2}} ds,$$

where $p = 2\gamma$ and $\beta \in (\frac{1}{p}, \frac{1}{2})$. From the Hausdorff-Young inequality and (N2) we get

$$\mathbf{E} \int_0^{t_1} \|Y_n(s)\|^p \le K_4 \left(\int_0^{t_1} t^{-2\beta} dt \right)^{\frac{p}{2}} (1 + \mathbf{E} \sup_{t \in [0,t_1]} \|u_t^n\|_C^p).$$

Furthermore, from (2.1) and (3.2) we have

$$\mathbf{E} \sup_{t \in [0,t_1]} \|f(u_t^n)\|^2 + \mathbf{E} \sup_{t \in [0,t_1]} \|g(u_t^n)\|^p \le C(p).$$

For $r > 0$, we consider set $K(r)$:

$$\begin{aligned} K(r) := \{ & S(t)(\phi^n(0) - g(\phi^n)) \\ & + \int_0^t AS(t-s)z(s)\, ds + (G_1 \Psi)(t) + \frac{\sin(\beta\pi)}{\pi} (G_\beta \eta)(t) : \\ & \|\phi\|_C \le r, \|z\|_{C([0,t_1],V)} \le r, \|\Psi\|_{L^2((0,t_1),H)} \le r, \|\eta\|_{L^{2\gamma}((0,t_1),H)} \le r \}. \end{aligned}$$

Combining (3.2), the conditions on g, and the Lemmas 3.2 and 3.3, we conclude that $K(r)$ is a compact subset of $C([0,t_1],H)$. Therefore, for all $\varepsilon > 0$ one can find $r > 0$ such that for all n

$$\begin{aligned} P\{ & \|\phi^n\|_C > r, \|g(u_s^n)\|_{C([0,t_1],V)} > r, \\ & \|f(u_s^n)\|_{L^2((0,t_1),H)} > r, \|Y_n\|_{L^{2\gamma}((0,t_1),H)} > r \} < \varepsilon. \end{aligned}$$

In particular, for all $\varepsilon > 0$ there exists a compact set R_ε such that

$$P\{u^n \notin R_\varepsilon\} < 2\varepsilon.$$

This way, the sequence of measures

$$\mu_n(A) := P\{u^n \in A\}$$

is tight in $C([0,t_1],H)$, and by Prokhorov's theorem this set is weakly compact. Therefore, the Skorohod embedding theorem [26] guarantees that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with Wiener process $\tilde{W}(t)$ and filtration $\tilde{\mathcal{F}}_t$ such that the sequence $\tilde{u}^n(t)$ has the same distribution as u^n, and it converges with probability 1 to some $\tilde{u}$ in $C([0,t_1],H)$. Since $\tilde{u}^n$ and u^n have the same distributions, we have

$$\begin{aligned} \tilde{u}^n \to \tilde{u} & \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, V), \\ f(\tilde{u^n}) \to \tilde{F} & \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, H), \\ \sigma(\tilde{u^n}) \to \tilde{\Sigma} & \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, L_2^0). \end{aligned}$$

On the other hand $\tilde{u}^n$ converges strongly to $\tilde{u}$ in $C(\tilde{\Omega}_{t_1}, H)$ with probability 1, hence $\tilde{u}^n(t,\omega)$ converges pointwise to $\tilde{u}(t,\omega)$.

Recalling that $\tilde{\Omega}$ is, in fact, the interval $[0,1]$ with Lebesgue measure. Then [15, Lemma 1.3] implies

$$f(\tilde{u^n}) \to f(\tilde{u}) \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, H),$$

$$\sigma(\tilde{u^n}) \to \sigma(\tilde{u}) \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, L_2^0),$$

$$g(\tilde{u^n}) \to g(\tilde{u}) \text{ weakly in } L^2(\tilde{\Omega}_{t_1}, H).$$

Using the the properties of stochastic integral, we deduce, that $\int_0^t \sigma(u_s)\,dW(s)$ is a continuous linear operator from the Banach space $L^2(\Omega_{t_1}, L_2^0)$ to the Banach space $L^2(\Omega_T, H)$. Therefore, it is continuous with respect to weak convergence, and

$$\int_0^t \Big(\sigma(\tilde{u}_s^n)d\tilde{W}^n(s), v\Big) \to \int_0^t \Big(\sigma(\tilde{u}_s)d\tilde{W}(s), v\Big) \text{ as } n \to \infty.$$

Thus, by Vitali theorem, we deduce that

$$\mathbf{E} \sup_{t\in[0,T]} \|\tilde{u}^n(t) - \tilde{u}(t)\|^2 \to 0 \text{ as } n \to \infty.$$

For all $v \in V$ and $\psi \in L^\infty([0,t_1] \times \Omega)$ using the Fubini theorem we have

$$\mathbf{E}\left(\int_0^t (\tilde{u}(s) - g(\tilde{u}_s), \psi(s)v)\,ds\right) = \lim_{n\to\infty} \mathbf{E}\left(\int_0^t (\tilde{u}^n(s) - g^n(\tilde{u}_s^n), \psi(s)v)\,ds\right)$$
$$= \lim_{n\to\infty} \mathbf{E}\Big[\int_0^t (\tilde{\phi}^n(0) - g^n(\tilde{\phi}^n)), \psi(t)v)\,ds + \int_0^t \int_0^s \big\langle A^n(\tilde{u}^n(\tau)), \psi(s)v\big\rangle\, d\tau\, ds$$
$$+ \int_0^t \int_0^s (f^n(\tilde{u}_\tau^n), \psi(s)v) d\tau\, ds + \int_0^t \int_0^s (\sigma^n(\tilde{u}_\tau^n), \psi(s)v) d\tilde{W}^n(\tau)\, ds\Big]$$
$$= \mathbf{E}\Big(\int_0^t (\tilde{u}(0) - g(\tilde{\phi}) + \int_0^s A(\tilde{u}(\tau))\,d\tau$$
$$+ \int_0^s f(\tilde{u}_\tau)\,d\tau + \int_0^s \sigma(\tilde{u}_\tau) d\tilde{W}(\tau), \psi(s)v)\,ds\Big).$$

Therefore, from Theorem 2.1 and the Yamada-Watanabe Theorem [32], we obtain the continuous dependence on the initial data. Theorem is proved. □

4. Conclusion

We derived the conditions on the uniqueness and continuous dependence on the initial data for weak solutions of neutral type stochastic functional-differential equations in Hilbert spaces.

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