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MODELING PHYSICAL PROCESSES USING COMSOL MULTIPHYSICS

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The Modified Korteweg-De Vries equation models a variety of nonlinear phenomena, including ion acoustic waves in plasmas, and shallow water waves. In this research the mKdV equation shall be simulated using Comsol Multiphysics, computer simulation environment where this partial differential equation can be modeled and visualized. Generally, this equation looks the next way:

$$u_t - au^2 u_x + u_{xxx} = 0\,. \tag{1}$$

The parameter a can be considered as any real number, where the commonly used values are $a = \pm 1$ or $a = \pm 6$.

The function $u(x,t)$ represents the water's free surface in non-dimensional variables. The nonlinear KdV equation gives a large variety of solutions. The solutions propagate at speed c while retaining its identity. We usually introduce the new wave variable $\xi = x - ct$, so that $u(x,t) = u(\xi)$.

The derivative ut characterizes the time evolution of the wave propagating in one direction, the nonlinear term uux describes the steepening of the wave, and the linear term $uxxx$ accounts for the spreading or dispersion of the wave. [1]

Obtaining analytical solution of Modified Korteweg-De Vries equation

Integrating once (1) we find, where constant of integration is taken to be zero:

$$-cu + \frac{a}{3}u^3 + u'' = 0. \tag{2}$$

Then we use cosine method to find the solution of the mKdV equation:

$$u = \lambda\, cos^{\beta}(\mu\xi), \tag{3}$$

$$u^3 = \lambda^3\, cos^{3\beta}(\mu\xi), \tag{4}$$

$$u'' = -\mu^2\beta^2\lambda\, cos(\mu\xi) + \mu^2\lambda\beta(\beta-1)cos^{\beta-2}(\mu\xi), \tag{5}$$

$$-c\lambda\, cos^{\beta}(\mu\xi) + \frac{a}{3}\lambda^3\, cos^{3\beta}(\mu\xi) - \mu^2\beta^2\lambda\, cos^{\beta}(\mu\xi) +$$

$$+\mu^2\lambda\beta(\beta-1)cos^{\beta-2}(\mu\xi) = 0\,. \tag{6}$$

We balance the exponents of the cosine functions:

$$3\beta = \beta - 2, \tag{7}$$

$$\beta = -1, \tag{8}$$

which leads to:

$$\mu^2\beta^2\lambda = -c\lambda, \tag{9}$$

$$\mu = \sqrt{-c}\,, \tag{10}$$

$$\frac{a}{3}\lambda^3 = -\mu^2\lambda\,\beta(\beta-1), \tag{11}$$

$$\frac{a}{3}\lambda^2 = -2\sqrt{-c}^{\,2}, \tag{12}$$

$$\lambda = \sqrt{\frac{6c}{a}}. \tag{13}$$

So, we can conclude that we found $= -1$, $\lambda = \sqrt{\frac{6c}{a}}$, $\mu = \sqrt{-c}$. Now, we can substitute all the results into (3).

$$u(x,t) = \sqrt{\frac{6c}{a}}\cos^{-1}\sqrt{-c}(x-ct), \tag{14}$$

$$u(x,t) = \sqrt{\frac{6c}{a}}\sec\sqrt{-c}(x-ct). \tag{15}$$

This in turn gives the periodic solutions for $c < 0,\ a < 0$:

$$u(x,t) = \sqrt{\frac{6c}{a}}\operatorname{sech}\sqrt{-c}(x-ct). \tag{16}$$

Equation (16) is also a travelling wave solution of the KdV equation. It is also seen from Equation (16) that the amplitude of the wave is directly proportional to its speed c, and this, in turn, means that the higher the wave, the faster it moves.

We consider initial value of solution when time is equal to zero:

$$u(x,0) = \sqrt{\frac{6c}{a}}\operatorname{sech}\sqrt{-c}(x), \tag{17}$$

and consider constants as $c = -1\ and\ a = -1$.

$$u(x,0) = \sqrt{6}\operatorname{sech}(x). \tag{18}$$

Simulation of Modified Korteweg-De Vries equation using Comsol Multiphysics

In Comsol Multyphysics, we modify the standard KdV equation of the form (1) to the new one: [2]

$$u_{1t} - a{u_1}^2 u_{1x} + u_{1_{xxx}} = 0. \tag{19}$$

We move $-au^2u_x$ to the right part of equation then:

$$u_{1t} + u_{1xxx} = a{u_1}^2 u_{1x} \,. \tag{20}$$

Then we insert new function and derivate it by x:

$$u_2 = u_{1xx}, \tag{21}$$

$$u_{2x} = u_{1xxx}. \tag{22}$$

Then we substitute (22) into (20)

$$u_{1t} + u_{2x} = a{u_1}^2 u_{1x} \,. \tag{23}$$

Then we get a system of equations from equations (23) and (21):

$$\begin{cases} u_{1t} + u_{2x} = a{u_1}^2 u_{1x} \\ u_2 = u_{1xx} \end{cases}. \tag{24}$$

With a boundary condition:

$$u_1(-8, t) = u_1(8, t) \,. \tag{25}$$

For an initial condition we use equation (24) and (18) with $t = 0$, *and* when $a = -1$ and $c = -1$:

$$u_1(x, 0) = \sqrt{6}\,\mathrm{sech}(x), \tag{26}$$

$$u_2(x, 0) = u_{1xx}(x, 0) \,. \tag{27}$$

Comsol Multiphysics Modified Korteweg-De Vries equation simulation results:

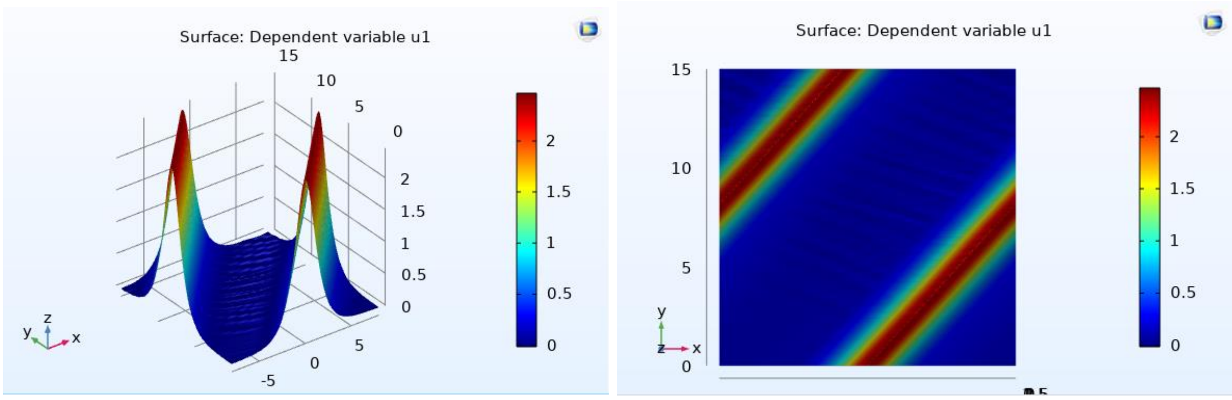

Figure 1. Surface plot for dependent variable u_1 by x, y, z coordinates and x, y for an initial condition $u(x, 0) = \sqrt{6}\,\mathrm{sech}(x)$, when $a = -1$ and $c = -1$.

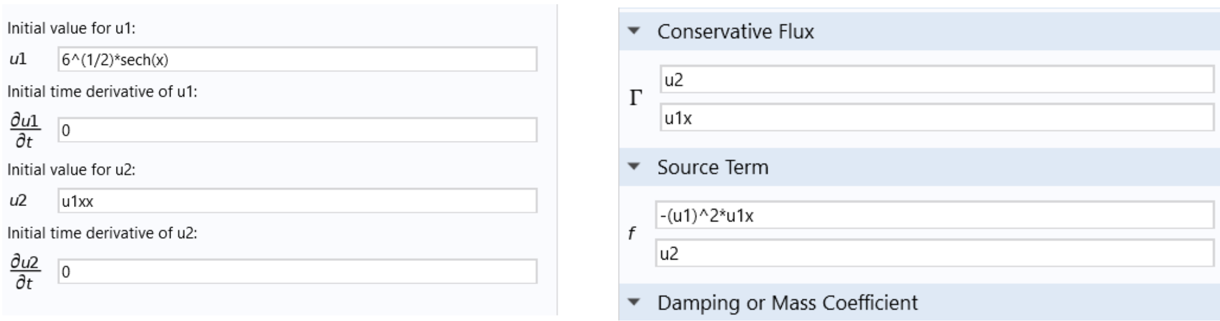

Figure 2. Initial values for the Modified KdV equation and its interpretation in Comsol Multiphysics.

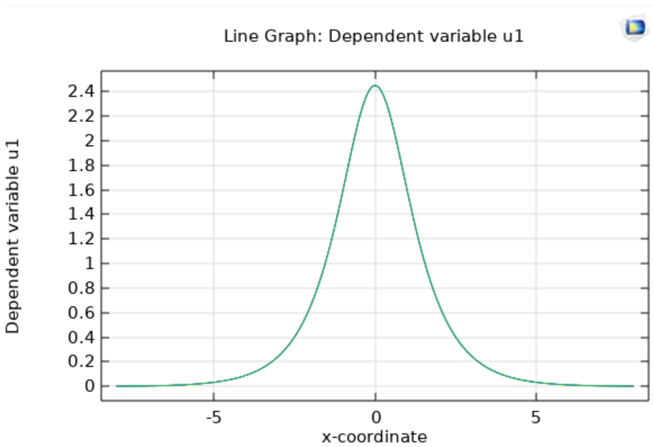

Figure 3. Line Graph for dependent variable u_1 for an initial condition $u(x, 0) = \sqrt{6}\,\mathrm{sech}(x)$, when $a = -1$ and $c = -1$.

In this paper, I made an example of how Modified Kortweg-De Vries equation can be simulated in Comsol Multiphysics. Discussing the results of an initial condition $u(x, 0) = \sqrt{6}\, sech(x)$ we can see a number of clear peaks with boundary condition $u_1(-8, t) = u_1(8, t)$. The periodic nature of waves can be seen from the results expressed in the figures 1 and 3. In the Figure 2 was represented a window of initial condition input and general form of PD equation representation in Comsol Multiphysics.

Comsol Multiphysics has a lot of benefits like user friendly interface, a rich physics opportunities and real world precision in simulations. This makes the analysis of nonlinear partial differential equations more accurate and improves our studies even more.

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References

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