УДК 517.51

WEIGHTED ESTIMATES FOR DISCRETE ITERATED HARDY OPERATOR

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Let $0 < r < \infty$ and $0 < p \le q < \infty$. Let $\{u_i\}_{i=1}^{\infty}$, $\{\vartheta_i\}_{i=1}^{\infty}$, $\{\omega_i\}_{i=1}^{\infty}$ be non-negative sequences of real numbers. Denote by $l_{p,\vartheta}$ a space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that $\|f\|_{l_{p,\vartheta}} = \left(\sum_{i=1}^{\infty} \vartheta_i |f_i|^p\right)^{\frac{1}{p}} < \infty$, where $1 \le p < \infty$.

We consider the following iterated discrete Hardy inequalities

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{k=1}^{i} \left|\sum_{j=k}^{i} f_j\right|^r \omega_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \le C_1 \left(\sum_{i=1}^{\infty} \vartheta_i |f_i|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,\vartheta} \tag{1}$$

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{k=i}^{\infty} \left|\sum_{j=i}^{k} f_j\right|^r \omega_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \le C_2 \left(\sum_{i=1}^{\infty} \vartheta_i |f_i|^p\right)^{\frac{1}{p}}, \qquad \forall f \in l_{p,\vartheta} \tag{2}$$

The Hardy inequalities, which have been studied in depth since the early XX century, have their applications in many areas of mathematics. Hardy type inequalities are an important tool in solving problems of mathematics and mathematical physics and they are widely used in the theory of integral and differential equations, in non- linear analysis, in the spectral theory of elliptic type.

In recent years, one of the most intensively studied topics in the theory of Hardy inequalities has been the evaluation of iterated operators. The reason for this is the wide application of these inequalities in the study of boundedness properties of operators from a Lebesgue weighted space to a local Morrey type space, as well as in the study of bilinear operators in weighted Lebesgue spaces.

An inequality involving an iteration of discrete, continuous Hardy operator is considered difficult to estimate since it has three independent weights and three parameters with different ratios. Nevertheless, many papers are devoted to this type of inequality. Characterization of continuous iterated Hardy inequalities were obtained in works [1]-[5]. Compared to the continuous case the discrete analogue of the Hardy iterated inequality is studied very little. In this direction, we can note the works of Gogatishvilli A., Krepela M., Rastislav O., Pick L. [6] and Oinarov R., Temirhanova A.M., Omarbaeva B.K. [7],[8].

The aim of this work is to obtain necessary and sufficient conditions for the fulfillment of the discrete iterated Hardy inequalities (1) and (2) for cases: $0 < p \le 1$, $0 < p \le \min\{q, r\} < \infty$; $0 < r < 1 < p \le q < \infty$.

Let

$$A^{+}_{z,\infty} = \sup_{z \le k \le \infty} \left(\sum_{j=k}^{\infty} \omega_j\right)^{\frac{1}{r}} \vartheta_k^{-\frac{1}{p}}, \quad A^{-}_{1,z} = \sup_{1 \le k \le z} \left(\sum_{j=1}^{k} \omega_j\right)^{\frac{1}{r}} \vartheta_k^{-\frac{1}{p}},$$

$$B_{z,\infty}^{+}=\left(\sum_{k=z}^{\infty}\left(\sum_{j=k}^{\infty}\omega_j\right)^{\frac{p}{p-r}}\left(\sum_{j=z}^{k}\vartheta_j^{1-p'}\right)^{\frac{p(r-1)}{p-r}}\vartheta_k^{1-p'}\right)^{\frac{p-r}{pr}},$$

$$B_{1,z}^{-}=\left(\sum_{k=1}^{z}\left(\sum_{j=1}^{k}\omega_j\right)^{\frac{p}{p-r}}\left(\sum_{j=k}^{z}\vartheta_j^{1-p'}\right)^{\frac{p(r-1)}{p-r}}\vartheta_k^{1-p'}\right)^{\frac{p-r}{pr}},$$

$$U_z^{+}=\left(\sum_{i=1}^{z}u_i\right)^{\frac{1}{q}},\quad U_z^{-}=\left(\sum_{i=z}^{\infty}u_i\right)^{\frac{1}{q}},$$

where $z\in\mathrm{N}$.

Our main result reads as follows:

Theorem 1. The inequality (1) holds if and only if

(i) $E=\sup\limits_{z\in N}U_z^{-}A_{1,z}^{-}<\infty$ for $0<p\le 1$, $0<p\le\min\{q,r\}<\infty$;

(ii) $E=\sup\limits_{z\in N}U_z^{-}B_{1,z}^{-}<\infty$ for $0<r<1<p\le q<\infty$.

Moreover $E\approx C_1$. Here, C_1 is the smallest constant that satisfies the inequality (1).

Theorem 2. The inequality (2) holds if and only if

(i) $E=\sup\limits_{z\in N}U_z^{+}A_{z,\infty}^{+}<\infty$ for $0<p\le 1$, $0<p\le\min\{q,r\}<\infty$;

(ii) $E=\sup\limits_{z\in N}U_z^{+}B_{z,\infty}^{+}<\infty$ for $0<r<1<p\le q<\infty$.

Moreover, $E\approx C_2$, where C_2 is the smallest constant that satisfies the inequality (2).

The notation $A<<B$ means that there exists a constant $c>0$ depending only on parameters p, r, q, such that the inequality $A\le cB$ is fulfilled. We write $A\approx B$ if a two-way estimation $A<<B<<A$ holds.

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