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THE CAUCHY TYPE PROBLEMS FOR q-DIFFERENTIAL EQUATIONS WITH THE RIEMANN-LIOUVILLE FRACTIONAL q-DERIVATIVES

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We recall some elements of q-calculus, for more information see e.g. the books [1], [2] and [3]. Throughout this paper, we assume that $0 < q < 1$ and $0 \leq a < b < \infty$.

Let $\alpha \in R$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1-q^2}{1-q},$$

where $\lim_{q\to 1}\frac{1-q^2}{1-q}=\alpha$.

We introduce for $k \in N$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n}\left(1-q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty}(a;q)_q^n \text{, и } (a;q)_n = \frac{(a;q)_\infty}{\left(q^\alpha a;q\right)_\infty}.$$

The q-analogue of the power function $(a-b)_q^\alpha$ is defined by

$$(a-b)_q^\alpha := a^\alpha \frac{\left(\frac{a}{b};q\right)_\infty}{\left(q^\alpha \frac{a}{b};q\right)_\infty}.$$

Notice that $(a-b)_q^\alpha = a^\alpha\left(\frac{a}{b};q\right)_\alpha$.

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{\left(q^x;q\right)_\infty}(1-q)^{1-x},$$

for any $x>0$.

The q-integral (or Jackson integral) $\int_a^b f(x)d_q x$ is defined by

$$\int_0^a f(x)d_q x := (1-q)a\sum_{m=0}^{\infty} q^m f\left(aq^m\right)$$

for $a=0$.

Definition 1. Let $\Omega=[a,b]\,(-\infty<a<b<\infty)$ be a finite interval on the real axis *R*. The *Riemann-Liouville fractional integrals* $\left(I_{0+,q}^\alpha\right)(x)$ of order $\alpha \in C$ $(R(\alpha)>0)$ are defined by

$$\left(I_{0+,q}^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)_q^{\alpha-1} f(t)d_q t$$

respectively. Here $\Gamma_q(\alpha)$ is the q-Gamma function.

Definition 2. The Riemann-Liouville fractional q-derivative $D_{q,a+}^\alpha f$ of order $\alpha>0$ is defined by

$$\left(D_{q,a+}^\alpha f\right)(x) := \left(D_{q,a+}^{[\alpha]} I_{q,a+}^{[\alpha]-\alpha} f\right)(x).$$

We consider the Cauchy type problem in the following form:

$$\left(D_{0+,q}^{\alpha}y\right)(x)-\lambda y(x)=f(x)\ \left(0\le a<x\le b;\alpha>0;\lambda\in R\right), \tag{1}$$

$$\left(D_{0+,q}^{\alpha-k}y\right)(0+)=b_k \qquad (b_k\in R; k=1,...,n=-[-\alpha]). \tag{2}$$

Theorem. Let $\alpha>0$, $n=-[-\alpha]$ and $\gamma(0\le\gamma<1)$ be such that $\gamma\ge n-\alpha$. Also let $\lambda\in R$. If $f\in C_{q,\gamma}[a,b]$, then the Cauchy type problem (1)-(2) has a unique solution $y(x)\in C_{q,n-\alpha,\gamma}^{\alpha}[a,b]$ and this solution is given by

$$y(x)=\sum_{j=1}^{n}b_j t^{\alpha-j}E_{\alpha,\alpha-j+1;q}\left[(\lambda t)^{\alpha}\right]+\int_0^x (x-qt)_q^{\alpha-1}E_{\alpha,\alpha;q}\left[\lambda(x-qt)_q^{\alpha}\right]f(t)d_q t$$

where $E_{\alpha,\beta;q}\left(z^{\alpha}\right):=\sum_{k=0}^{\infty}\frac{z^{\alpha k}}{\Gamma_q(\alpha+\beta)}$.

Literature

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