# Discrete symmetry transformations on non-abelian gauge fields

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#### Abstract

All gauge bosons of a non-abelian gauge theory do not transform the same way under the discrete transformations of time-reversal and charge-conjugation. Moreover, the transformations rules depend on how the generators are chosen. We show how well-defined rules pertain only to specific choices of generators, and then show how unified rules can be constructed, using matrix forms of the gauge bosons, which are completely independent of the choice of generators.

# 1 Introduction

The extended Lorentz group (ELG) contains the discrete symmetries of space-inversion or parity, and time-reversal. In addition, there is an external automorphism of the Lorentz group in the form of charge conjugation. The transformation of the photon field under these discrete symmetry operations are well-known and well-discussed [1]. In non-abelian gauge theories, some special features appear in the transformation of the gauge fields which are absent in the abelian theory of quantum electrodynamics. In this note, we clarify the transformations of the gauge fields, as well as the associated field strength tensors.

Needless to say, this is not a new issue. There exist discussions on these topics in the literature. We want to improve on the previous discussion in a few ways. First, we want to show explicitly that the transformation properties of all gauge bosons are not the same under time-reversal and charge-conjugation. This has been pointed out in the context of classical solutions of Yang-Mills theories [2], while in the context of quantum field operators this is usually implicit [3–5]. Second, we point out exactly which property of the generators is important for writing the transformation rules, irrespective of any particular choice of generators. Third,

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we to cast the transformation rules in a way that would be valid irrespective of any special property, like hermiticity, of the generators.

We set up our notation in the rest of this Introduction. The gauge fields appear in two different combinations in a non-abelian gauge theory. First, they appear through the gauge-covariant derivative,

$$D_{\mu} = \partial_{\mu} + igR^{a}A^{a}_{\mu}, \qquad (1)$$

where  $R^a$ 's denote the generators of the gauge group and g is the gauge coupling constant. This combination governs the coupling of the gauge fields to all other fields in the theory. Second, the pure gauge Lagrangian contains the field-strength tensor, which is the combination

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,, \tag{2}$$

where  $f^{abc}$ 's are the structure constants, defined by

$$[R^a, R^b] = if^{abc}R^c. (3)$$

Our aim is to discuss the transformation of the gauge fields  $A^a_\mu$  and the field-strength tensors  $F^a_{\mu\nu}$  under the discrete transformations of parity, time-reversal and charge-conjugation.

# 2 Parity

Parity transformation is the most straightforward among the three, so we discuss it first. Under parity, the spacetime derivatives transform as follows:

$$\partial_0 \xrightarrow{P} \partial_0, \qquad \partial_i \xrightarrow{P} -\partial_i.$$
 (4)

Therefore, the other term in the expression for the covariant derivative must transform in the same way in order for a theory to be parity invariant. Since parity is a linear operator that does not affect the generators, we conclude that

$$A_0^a(t,\vec{x}) \xrightarrow{P} A_0^a(t,-\vec{x}), \qquad A_i^a(t,\vec{x}) \xrightarrow{P} -A_i^a(t,-\vec{x}).$$
 (5)

This is similar to the transformations obtained for the photon field in QED: no new feature appears.

## 3 Time-reversal

New features appear for the time-reversal transformation, which we will denote by  $\mathscr{T}$ . Now the derivatives change as

$$\partial_0 \xrightarrow{T} \widetilde{\partial}_0 = -\partial_0, \qquad \partial_i \xrightarrow{T} \widetilde{\partial}_i = \partial_i.$$
 (6)

To write how the covarant derivative transforms, we remember that time-reversal is an antilinear operator, so that all coefficients should be complex conjugated. Thus,

$$\mathscr{T}D_{\mu}\mathscr{T}^{-1} = \widetilde{\partial}_{\mu} - ig(R^a)^* \widetilde{A}^a_{\mu}, \qquad (7)$$

where the tilde notation on the gauge fields is a shorthand of the time-reversed fields:

$$\widetilde{A}_{\mu}(t,\vec{x}) = A_{\mu}(-t,\vec{x}). \tag{8}$$

This means that time-reversal invariance would be obtained if the gauge fields transform in the following way:

$$\mathscr{T}A_0^a\mathscr{T}^{-1} = \begin{cases} +\tilde{A}_0^a & \text{if } R^a \text{ is real,} \\ -\tilde{A}_0^a & \text{if } R^a \text{ is imaginary,} \end{cases}$$
(9a)

$$\mathscr{T}A_i^a\mathscr{T}^{-1} = \begin{cases} -\tilde{A}_i^a & \text{if } R^a \text{ is real,} \\ +\tilde{A}_i^a & \text{if } R^a \text{ is imaginary.} \end{cases}$$
(9b)

This feature, that different gauge bosons transform differently [6,7], is not present in an abelian theory. Because of the different transformation properties, any arbitrary combination of generators may not have a well-defined transformation property under time-reversal.

Note that we have never used any other property of the generators except their reality. For example, Eq. (9) is valid irrespective of any specific hermiticity property of the generators. In fact, two different kinds of transformation rules are required for internal consistency. To see what we mean, consider two gauge bosons  $W^{\pm}_{\mu}$  of an SU(2) gauge group, which couple to the ladder generators  $R^{\pm} \equiv (R^1 \pm i R^2)/\sqrt{2}$ , in the conventional choice in which  $R^3$  is taken to be diagonal (and therefore real), whereas  $R^1$  is real and  $R^2$  is purely imaginary. Then, using Eq. (9), we get

$$\mathscr{T}W_0^{\pm}\mathscr{T}^{-1} = \frac{1}{\sqrt{2}}\mathscr{T}(W_0^1 \pm iW_0^2)\mathscr{T}^{-1} = \frac{1}{\sqrt{2}}(\widetilde{W}_0^1 \pm (-i)(-\widetilde{W}_0^2)) = \widetilde{W}_0^{\pm}, \tag{10}$$

which is the appropriate transformation rule since the associated generators are real.

# 4 Charge-conjugation

Let us now turn to charge-conjugation. It is a linear operator, like parity. However, there is one way that it differs from the previous two. If we apply charge-conjugation on a charged gauge field  $W^+$ , we expect to obtain  $W^-$ . As we know,  $W^+_{\mu}$  and  $W^-_{\mu}$  are different 4-vectors, i.e., they belong to different representations of the ELG. That is why charge-conjugation is called an outer automorphism.

In order to deal with it, it is not enough to look at the covariant derivative. We need to consider the entire interaction term with the gauge field. The covariant derivative tells us that we can write the interaction term as

$$\mathcal{L}_{\text{int}} = -J_{\mu}^{a} A^{a\mu} \,, \tag{11}$$

where  $J^a_{\mu}$  is the current. We need its transformation property in order to discuss the transformation property of the gauge fields.

Let us first consider the current of a multiplet of scalar fields:

$$J_{\mu}^{a} = i \left( \phi^{\dagger} R^{a} \partial_{\mu} \phi - (\partial_{\mu} \phi^{\dagger}) R^{a} \phi \right)$$
$$= i \left( \phi_{\alpha}^{\dagger} (R^{a})_{\alpha\beta} \partial_{\mu} \phi_{\beta} - (\partial_{\mu} \phi_{\alpha}^{\dagger}) (R^{a})_{\alpha\beta} \phi_{\beta} \right), \tag{12}$$

where in the last step, we have explicitly shown the indices  $\alpha, \beta$ , which refer to different components of the multiplet  $\phi$ . Under charge-conjugation, any scalar field would transform as follows:

$$\mathscr{C}\phi\mathscr{C}^{-1} = \phi^{\dagger} \,. \tag{13}$$

Thus,

$$\mathscr{C}J_{\mu}^{a}\mathscr{C}^{-1} = \mathrm{i}\left(\phi_{\alpha}(R^{a})_{\alpha\beta}\partial_{\mu}\phi_{\beta}^{\dagger} - (\partial_{\mu}\phi_{\alpha})(R^{a})_{\alpha\beta}\phi_{\beta}^{\dagger}\right)$$

$$= \mathrm{i}\left(-\phi_{\beta}^{\dagger}(R^{a})_{\alpha\beta}\partial_{\mu}\phi_{\alpha} + (\partial_{\mu}\phi_{\beta}^{\dagger})(R^{a})_{\alpha\beta}\phi_{\alpha}\right)$$

$$= \mathrm{i}\left(\phi^{\dagger}(-R^{a})^{\top}\partial_{\mu}\phi - (\partial_{\mu}\phi^{\dagger})(-R^{a})^{\top}\phi\right). \tag{14}$$

This shows that the representation matrices in the conjugate representation are  $-(R^a)^{\top}$ . Note that the algebra of Eq. (3) implies that

$$[(-R^a)^{\top}, (-R^b)^{\top}] = if^{abc}(-R^c)^{\top}$$
(15)

for any choice of generators, so the matrices  $-(R^a)^{\top}$  constitute a representation of the same algebra.

The same conclusion can be reached if we consider the current of fermion fields. In this case, the current is given by

$$J^a_\mu = \bar{\psi}\gamma_\mu R^a \psi = (\psi^\dagger)_{\alpha A} (\gamma_0 \gamma_\mu)_{AB} (R^a)_{\alpha \beta} \psi_{\beta B} \,. \tag{16}$$

In the last step, we have shown all indices, including the Dirac indices in the form of A, B. The charge conjugation rule for fermion fields are as follows [8]:

$$\mathscr{C}\psi_A\mathscr{C}^{-1} = (\gamma_0 C)_{AB}(\psi^{\dagger})_B, \tag{17}$$

which also implies

$$\mathscr{C}\psi_A^{\dagger}\mathscr{C}^{-1} = \psi_B(C^{-1}\gamma_0)_{BA}. \tag{18}$$

Therefore,

$$\mathscr{C}J^{a}_{\mu}\mathscr{C}^{-1} = \mathscr{C}(\psi^{\dagger})_{\alpha A}\mathscr{C}^{-1}(\gamma_{0}\gamma_{\mu})_{AB}(R^{a})_{\alpha\beta}\mathscr{C}\psi_{\beta B}\mathscr{C}^{-1}, \tag{19}$$

using the linearity of the transformation. Using Eq. (17) and Eq. (18) now, we obtain

$$\mathscr{C}J_{\mu}^{a}\mathscr{C}^{-1} = \psi_{\alpha E}(C^{-1}\gamma_{0})_{EA}(\gamma_{0}\gamma_{\mu})_{AB}(R^{a})_{\alpha\beta}(\gamma_{0}C)_{BF}(\psi^{\dagger})_{\beta F},$$

$$= -(\psi^{\dagger})_{\beta F}(\gamma_{0}C)_{BF}(R^{a})_{\alpha\beta}(\gamma_{0}\gamma_{\mu})_{AB}(C^{-1}\gamma_{0})_{EA}\psi_{\alpha E}. \tag{20}$$

In the last step, we have merely rearranged the terms. The minus sign comes because of anticommutation of fermion fields. Now we can get rid of the Dirac indices to write

$$\mathscr{C}J^{a}_{\mu}\mathscr{C}^{-1} = -(\psi^{\dagger})_{\beta}(\gamma_{0}C)^{\top}(R^{a})_{\alpha\beta}(\gamma_{0}\gamma_{\mu})^{\top}(C^{-1}\gamma_{0})^{\top}\psi_{\alpha}. \tag{21}$$

The definition of the matrix C implies

$$C^{-1}\gamma_{\mu}C = -\gamma_{\mu}^{\top}, \qquad C^{\top} = -C.$$
 (22)

Using these relations, it is easily seen that

$$\mathscr{C}J_{\mu}^{a}\mathscr{C}^{-1} = \bar{\psi}\gamma_{\mu}(-R^{a})^{\top}\psi. \tag{23}$$

Once again, it shows that the relevant representation matrices are  $(-R^a)^{\top}$ .

So the combination given in Eq. (11) will be invariant under charge conjugation now demands

$$\mathscr{C}A_{\mu}^{a}\mathscr{C}^{-1} = \begin{cases} -A_{\mu}^{a} & \text{if } R^{a} \text{ is symmetric,} \\ +A_{\mu}^{a} & \text{if } R^{a} \text{ is antisymmetric.} \end{cases}$$
(24)

Note that it is only the symmetry property that matters. Facts like whether the generators are real, or hermitian, are of no importance.

The different transformations for the two types of gauge bosons is crucial in understanding what turns  $W^+$  into  $W^-$ , which are gauge bosons of SU(2) corresponding to the ladder generators. In the conventional basis in which the generator associated with  $W^3_{\mu}$  is taken as diagonal, the generators associated with  $W^{\pm}$  are not purely symmetric or antisymmetric. Therefore, we need to trace back to the generators which have well-defined symmetry properties in order to obtain the charge-conjugation properties of  $W^{\pm}$ .

$$\mathscr{C}W_{\mu}^{+}\mathscr{C}^{-1} = \frac{1}{\sqrt{2}}\mathscr{C}(W_{\mu}^{1} + iW_{\mu}^{2})\mathscr{C}^{-1} = \frac{1}{\sqrt{2}}\left(\mathscr{C}W_{\mu}^{1}\mathscr{C}^{-1} + i\mathscr{C}W_{\mu}^{2}\mathscr{C}^{-1}\right),\tag{25}$$

using the fact that charge-conjugation is a linear operator. In the representation of the SU(2) generators alluded above,  $W^1$  is associated with a generator that is real and symmetric, whereas  $W^2$  is associated with a generator that is imaginary and antisymmetric. Using Eq. (24) now, we obtain

$$\mathscr{C}W_{\mu}^{+}\mathscr{C}^{-1} = \frac{1}{\sqrt{2}} \left( -W_{\mu}^{1} + iW_{\mu}^{2} \right) = -W_{\mu}^{-}. \tag{26a}$$

Similarly, we will get

$$\mathscr{C}W_{\mu}^{-}\mathscr{C}^{-1} = -W_{\mu}^{+} \,. \tag{26b}$$

Note that it is not complex-conjugation of any number that ushers this change. In fact, the factor of i is unaffected since the operation is linear. It is the different transformation rules of the gauge bosons, as shown in Eq. (24), that is behind the change.

## 5 The matrix notation

Many authors prefer to use a matrix notation for the gauge fields. In other words, they consider the product of the generator and the gauge field and treat it like a matrix field. It will be instructive to consider the merits of such a notation. If we look at Eqs. (1) and (7), we see that the effect of time-reversal can be summarized by saying that

$$iR^a A^a_\mu \xrightarrow{T} (iR^a)^* \widetilde{A}^a_\mu$$
 (27)

Thus, if we write the covariant derivative in the form

$$D_{\mu} = \partial_{\mu} + g\mathbf{A}_{\mu} \tag{28}$$

by defining the matrix

$$\mathbf{A}_{\mu} \equiv iR^a A^a_{\mu} \,, \tag{29}$$

we can write the time-reversal transformation rule as

$$\mathscr{T} \mathbf{A}_0 \mathscr{T}^{-1} = -\widetilde{\mathbf{A}}_0^*, \qquad \mathscr{T} \mathbf{A}_i \mathscr{T}^{-1} = \widetilde{\mathbf{A}}_i^*, \tag{30}$$

where

$$\widetilde{A}_{\mu}^* = (iR^a)^* A_{\mu}^a \,. \tag{31}$$

Similarly, the effect of the charge-conjugation transformation can be summarized as

$$\mathscr{C} \boldsymbol{A}_{\mu} \mathscr{C}^{-1} = -\boldsymbol{A}_{\mu}^{\top} \equiv -(iR^{a})^{\top} A_{\mu}^{a}. \tag{32}$$

While these are the general rules that apply irrespective of the choice of generators, sometimes people write these rules in a way that apply to some specific choice that they have in mind. For example, if one chooses only hermitian generators, one can write

$$(R^a)^{\top} = (R^a)^*, \tag{33}$$

so that the charge-conjugation rule of Eq. (32) can be written as [4]

$$\mathscr{C} \boldsymbol{A}_{\mu} \mathscr{C}^{-1} = \boldsymbol{A}_{\mu}^{*}. \tag{34}$$

One can choose the hermitian generators to be either real symmetric or imaginary antisymmetric. We can use these properties to write

$$(R^a)^* = \begin{cases} R^a & \text{if } R^a \text{ is real symmetric,} \\ -R^a & \text{if } R^a \text{ is imaginary antisymmetric.} \end{cases}$$
(35)

Combining this with Eq. (9), we can write the time-reversal rule as

$$\mathscr{T}R^a A_0^a \mathscr{T}^{-1} = R^a A_0^a, \qquad \mathscr{T}R^a A_i^a \mathscr{T}^{-1} = -R^a A_i^a. \tag{36}$$

In terms of the matrices defined in Eq. (29), these equations can be written as

$$\mathscr{T} \mathbf{A}_0 \mathscr{T}^{-1} = -\widetilde{\mathbf{A}}_0, \qquad \mathscr{T} \mathbf{A}_i \mathscr{T}^{-1} = \widetilde{\mathbf{A}}_i,$$
 (37)

accounting for an extra negative sign coming from the fact that i  $\stackrel{T}{\longrightarrow}$  -i. However, it has to be remembered that Eqs. (34) and (37) are not valid for arbitrary choice of the generators. They use specific hermicity properties of the generators.

Previous authors have looked at discrete transformations of gauge field components in different contexts. Brandt [2] considered classical solutions and found the time-reversal transformations of gauge field components by treating them as gauge transformations. It appears that the rules are derived assuming a particular type of generators. Vafa and Witten [4] considered the quantum operators, and used the full matrix-valued gauge field  $\mathbf{A}_{\mu} \equiv \mathrm{i} R^a A_{\mu}^a \equiv S^a A_{\mu}^a$ . Their choice of the  $S^a$ 's is equivalent to Hermitian  $R^a$ 's, and they write the transformation rules under time-reversal and charge-conjugation in the form of Eq. (30) and Eq. (34) above. While these rules are consistent with their specific choice of generators, it should be emphasized that Eq. (34) is not valid for a general choice: instead, one should use Eq. (32). The same can be said about the joint transformation  $\mathscr{CT}[4,6,7]$ . On the other hand, Wang [9] wrote the time-reversal transformation rules in the same way as Vafa and Witten, but wrote the charge-conjugation transformation rule as  $\mathscr{C}A_0\mathscr{C}^{-1} = -A_0$ ,  $\mathscr{C}A_i\mathscr{C}^{-1} = A_i$ , which does not seem to correspond to the formulas we derived above.

#### 6 The field-strength tensor

We will now see how the components of the field-strength tensor transform under the discrete operations. The definition of the tensor was given in Eq. (2). For parity transformation, Eq. (5) gives the rules

$$F_{0i}^a(t,\vec{x}) \xrightarrow{P} -F_{0i}^a(t,-\vec{x}), \qquad F_{ij}^a(t,\vec{x}) \xrightarrow{P} F_{ij}^a(t,-\vec{x}),$$
 (38)

for the components of the field-strength tensor. As for the gauge fields themselves, these transformation rules are same as that for an abelian theory.

The situation changes for time-reversal and charge-conjugation. From Eq. (2), we see that

$$\mathcal{F}F_{0i}^{a}\mathcal{T}^{-1} = -\partial_{0}\mathcal{T}A_{i}^{a}\mathcal{T}^{-1} - \partial_{i}\mathcal{T}A_{0}^{a}\mathcal{T}^{-1} + g(f^{abc})^{*}\mathcal{T}A_{0}^{b}\mathcal{T}^{-1}\mathcal{T}A_{i}^{c}\mathcal{T}^{-1}, \qquad (39a)$$

$$\mathcal{F}F_{ij}^{a}\mathcal{T}^{-1} = \partial_{i}\mathcal{T}A_{j}^{a}\mathcal{T}^{-1} - \partial_{j}\mathcal{T}A_{i}^{a}\mathcal{T}^{-1} + g(f^{abc})^{*}\mathcal{T}A_{i}^{b}\mathcal{T}^{-1}\mathcal{T}A_{j}^{c}\mathcal{T}^{-1}. \qquad (39b)$$

$$\mathscr{T}F_{ij}^{a}\mathscr{T}^{-1} = \partial_{i}\mathscr{T}A_{j}^{a}\mathscr{T}^{-1} - \partial_{j}\mathscr{T}A_{i}^{a}\mathscr{T}^{-1} + g(f^{abc})^{*}\mathscr{T}A_{i}^{b}\mathscr{T}^{-1}\mathscr{T}A_{j}^{c}\mathscr{T}^{-1}.$$
(39b)

Suppose now each generator is either real or imaginary. That means that we have a relation of the form

$$(R^a)^* = \xi_{(a)}R^a$$
 (no sum on a), (40)

with each  $\xi_{(a)}$  either +1 or -1. Then, taking the complex conjugate of both sides of Eq. (3) and using the fact that the generators are linearly independent, we obtain the relation

$$\xi_{(a)}\xi_{(b)}\xi_{(c)} = -\frac{(f^{abc})^*}{f^{abc}}.$$
(41)

This means that, if an odd number of imaginary generators are involved in a particular  $f^{abc}$ , that  $f^{abc}$  will be real. Otherwise,  $f^{abc}$  will be imaginary. Upon complex-conjugation, the imaginary  $f^{abc}$ 's will change sign.

Let us now show how the different terms in  $F_{0i}^a$  would behave under time-reversal for different combinations of the reality properties of the generators.

The table shows that, irrespective of the reality property of  $R^b$  and  $R^c$ , one obtains the field-strength tensor to obtain a negative sign under time-reversal if  $R^a$  is real. The case of imaginary  $R^a$  can be covered similarly, and summarized into the rule

$$\mathscr{T}F_{0i}^{a}\mathscr{T}^{-1} = \begin{cases} -\tilde{F}_{0i}^{a} & \text{if } R^{a} \text{ is real,} \\ +\tilde{F}_{0i}^{a} & \text{if } R^{a} \text{ is imaginary.} \end{cases}$$
(43)

Once again we emphasize that these properties do not depend on any other characteristic of the generator. By an exactly similar analysis, we find

$$\mathscr{T}F_{ij}^{a}\mathscr{T}^{-1} = \begin{cases} +\tilde{F}_{ij}^{a} & \text{if } R^{a} \text{ is real,} \\ -\tilde{F}_{ij}^{a} & \text{if } R^{a} \text{ is imaginary.} \end{cases}$$
(44)

If, in analogy with Eq. (29), we define a matrix form of the field-strength tensor:

$$\mathbf{F}_{\mu\nu} = iR^a F^a_{\mu\nu} \,, \tag{45}$$

we can combine both real and imaginary generators in one single rule:

$$\mathscr{T} \mathbf{F}_{0i} \mathscr{T}^{-1} = \widetilde{\mathbf{F}}_{0i}, \qquad \mathscr{T} \mathbf{F}_{ij} \mathscr{T}^{-1} = -\widetilde{\mathbf{F}}_{ij}.$$
 (46)

We now perform a similar analysis for charge-conjugation. If each generator is either symmetric of antisymmetric, we can write

$$(R^a)^{\top} = \eta_{(a)} R^a \qquad \text{(no sum on } a), \tag{47}$$

where each  $\eta_{(a)}$  can be +1 and -1. Taking the transpose of Eq. (3) and comparing with the original, we now obtain

$$\eta_{(a)}\eta_{(b)}\eta_{(c)} = -1 \tag{48}$$

for each nonzero  $f^{abc}$ . Clearly, one cannot choose all generators to be symmetric, although all antisymmetric generators is a possibility. In general, there can be some symmetric and some antisymmetric generators. Since charge conjugation is a linear operator, it does not affect the structure constant, whether real or imaginary. Therefore, for symmetric  $R^a$ , we can make a chart of the possibilities that arise for different terms of  $F^{0i}$ :

Combining these results with similar ones obtained for antisymmetric  $R^a$ , we can write

$$\mathscr{C}F^{a}_{\mu\nu}\mathscr{C}^{-1} = \begin{cases} -F^{a}_{\mu\nu} & \text{if } R^{a} \text{ is symmetric,} \\ +F^{a}_{\mu\nu} & \text{if } R^{a} \text{ is antisymmetric.} \end{cases}$$
(50)

In the matrix notation, one can write

$$\mathscr{C} \boldsymbol{F}_{\mu\nu} \mathscr{C}^{-1} = -(\boldsymbol{F}_{\mu\nu})^{\top} . \tag{51}$$

Once again, it should be emphasized that the transformation rules, as given in Eqs. (46) and (51) are independent of the choice of the generators.

# 7 CPT, and other issues

Generators of a Lie group form a vector space. There are therefore innumerable ways that a basis set of generators can be defined. As we showed, for time-reversal and charge-conjugation, the transformation properties of the associated gauge bosons depend on the choice of generators. For time-reversal, gauge bosons corresponding to real and imaginary generators transform differently. If we choose a generator that has both real and imaginary parts, the corresponding gauge boson will not have a well-defined property under time-reversal. Similarly, for charge-conjugation, the difference is between symmetric and antisymmetric generators. If a generator is a combination of the two kinds, the corresponding gauge boson will not have a well-defined property under charge-conjugation.

For unitary groups, one can choose hermitian generators. These generators can be taken as purely real or purely imaginary, as for example the Pauli matrices for SU(2) or the Gell-Mann matrices for SU(3). However, it should be remembered that it is also possible to choose all real generators, by using the diagonal generators (which have to be real) and the combination of the other generators which act as ladder operators.

In the Introduction we said that the matrix notation hides the differences between the transformation properties of different gauge bosons. But there is an advantage as well, coming from the fact that the matrices  $A_{\mu}$  associated with all gauge bosons transform the same way, so that the transformation rules written in terms of these matrices are valid for arbitrary choices of generators, real or imaginary, symmetric or antisymmetric, or any mixture of any kind. This notation is therefore convenient for looking at the CPT transformation properties of the gauge bosons. Using the transformation rules for  $\mathcal{T}$ ,  $\mathcal{C}$  and  $\mathcal{P}$  separately, we get

$$\mathscr{CPT}\boldsymbol{A}_{0}(t,\vec{x})(\mathscr{CPT})^{-1} = -\mathscr{CP}\boldsymbol{A}_{0}^{*}(-t,\vec{x})(\mathscr{CP})^{-1} = -\mathscr{C}\boldsymbol{A}_{0}^{*}(-t,-\vec{x})\mathscr{C}^{-1}$$
$$= -(\boldsymbol{A}_{0}^{*})^{\top}(-t,-\vec{x}) = -\boldsymbol{A}_{0}^{\dagger}(-t,-\vec{x}). \tag{52}$$

Similarly, for the spatial components, we get

$$\mathscr{CPT}\boldsymbol{A}_{i}(t,\vec{x})(\mathscr{CPT})^{-1} = \mathscr{CP}\boldsymbol{A}_{i}^{*}(-t,\vec{x})(\mathscr{CP})^{-1} = -\mathscr{C}\boldsymbol{A}_{i}^{*}(-t,-\vec{x})\mathscr{C}^{-1}$$

$$= -(\boldsymbol{A}_{i}^{*})^{\top}(-t,-\vec{x}) = -\boldsymbol{A}_{i}^{\dagger}(-t,-\vec{x}). \tag{53}$$

In summary, then, we can write

$$\mathscr{CPT} \mathbf{A}_{\mu}(t, \vec{x}) (\mathscr{CPT})^{-1} = -\mathbf{A}_{\mu}^{\dagger}(-t, -\vec{x}).$$
 (54)

This is necessary for the CPT theorem to hold.

In conclusion, we demonstrated that different gauge bosons of a non-abelian gauge theory can transform differently under time-reversal and charge-conjugation, depending on the way the generators are chosen. We also showed how the matrix-valued gauge bosons can give the transformation rules in a way that does not depend on the choice of the generators. We also showed that these transformation rules, written in some specific ways using some specific choice of generators, are not valid in general, and indicated what the representation-independent rules should be.

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