

Primordial Cosmology of an Emergent-like Universe from Modified Gravity: Reconstruction and Phenomenology Optimization with a Genetic Algorithm

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General relativity has been the dominant paradigm theory of gravity for the last century, it manages to explain a plethora of astronomical and cosmological observations with high accuracy. A major challenge faced by general relativity is, that it predicts a singularity as the beginning of our universe. In this letter we shall explore a well known idea in the literature, that of the emergent universe and its viability according to Planck 2018 data and BBN constraints.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq, 11.25.-w

I. INTRODUCTION

Starting off there must be a discussion as to what an emergent universe model encompasses. In their paper George F R Ellis and Roy Maartens [182], describe an initial state of the universe in the infinite past, that inflates for a long period of time (in their analysis that inflationary phase lasted an infinite time), and after exiting the inflationary and reheating stages, emerges our own universe. That scenario has some very interesting and desirable characteristics. First off, it does not present a horizon problem as the initial state is an Einstein static universe, and thus matter and radiation had enough time to be in causal contact. Secondly, if the initial size is sufficient the universe never enters a Planckian epoch and consequently there is no need for quantum gravity, as the scale of interest is within the bounds of General Relativity. Lastly, it provides an elegant solution to the singularity problem mentioned in the abstract. The universe never had to experience a bouncing cosmology, as it inflated from its initial state, so no singularity occurred.

In their original work, the authors assumed a positive curvature universe and a scale factor constructed of exponential functions, to instigate their further research into emergent universe scenarios. We shall try to reconstruct a viable gravity model that produces a generalized version of their scale factor, all while assuming the mainstream proposition of a flat universe. The general form of the scale factor that will be used throughout this paper is:

$$a(t) = b \left(\exp \left(-\frac{\sqrt{2}t}{b} \right) + 1 \right)^n, \quad (1)$$

with b being positive and n negative, dimensionless quantities. It can be easily seen that for negative time, the scale factor of the universe is practically constant and approximately equal to ϵ , with ϵ being a small positive quantity. The scale factor is asymptotically equal to zero for negative time. To have the appropriate units, the argument t/b should be $t/(b^* \sqrt{\kappa})$, but as the analysis would be only phenomenological, the constant κ , shall be set to unity. A lot of ways to realize an emergent universe scenario have been explored in the literature, but in the scope of this paper we shall proceed with the reconstruction of an $f(R)$ gravity that gives rise to this particular scale factor. Albeit, the common factor used in such scenarios is of the form, $b(1 + e^{t/b})^n$, [183], these kinds of factor are not viable for matter-free reconstruction due to the emergence of imaginary expressions, which are unnatural. The inflationary phase of the universe ends naturally, with a finite number of e-folds, resulting in a subsequent reheating phase.

Before we delve deeper into the topic of emergent universe scenarios, it is important to address a common criticism leveled against them, which is related to the issue of fine-tuning. Although there are no scientific arguments that prevent the universe to have occurred as a result of precise tuning in its initial parameters, there exists a disdain in the cosmology circles regarding such hypotheses as unsatisfactory and fallacious, as they try to mask our ignorance about some yet discovered physical mechanism. Without getting ahead of ourselves, we must state that the current model reproduces the Planck 2018 observational data, for a wide range of its free parameters. In light of that fact, we want to draw the conclusion, that emergent universe models do not need fine tuning to be viable, and feasible models with explanatory power exist, it is just that they are a lot harder to find.

Inflation [1–4] is now put into focus from most current and forthcoming experiments. The aim is to pinpoint the B-modes directly experimentally, and this is the central focus in the stage 4 Cosmic Microwave Background (CMB)

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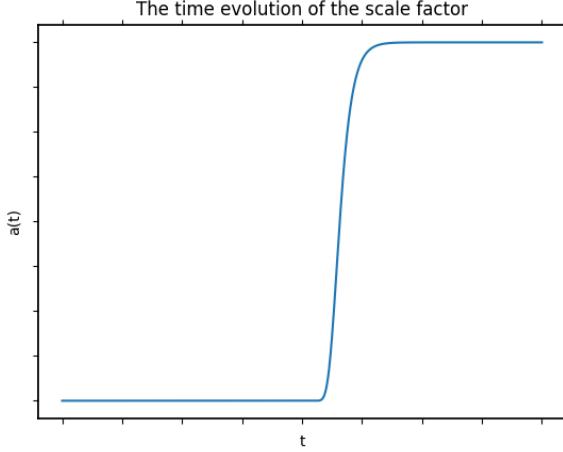


FIG. 1. The scale factor in terms of time , for $n=-60$ and $b=2000$

experiments [5, 6]. Apart from the B-modes, there are indirect ways to detect an inflationary era, for example the detection of a small (or negligible) anisotropy stochastic gravitational wave background [7–15]. These future gravitational wave experiments will directly probe the inflationary modes that became subhorizon after the inflationary era. Apart from the LIGO-Virgo successes and exciting observations, recently the NANOGrav collaboration confirmed the well anticipated stochastic gravitational wave detection [16], which was also confirmed by other pulsar timing arrays (PTA) experiments [17–19]. The scientific community was almost certain that the 2020 signal detection was not due to the actual pulsar red noise but due to a stochastic gravitational wave background, which was confirmed in 2023 due to the presence of Hellings-Downs correlations. After the 2023 NANOGrav announcement, a large stream of research articles were produced, trying to explain the signal, for the cosmological perspective see for example [20–49], see also [50–56] for earlier works in this perspective, and also [24, 25, 57] for the axion description. Although it is not certain whether the 2023 NANOGrav signal has a cosmological or astrophysical source, or even a combination of the two, there are theoretical hints toward the cosmological description. For the astrophysical description of the signal, see the recent review [58]. Currently there are some issues that render the cosmological description more plausible, although in principle the signal may be a hybrid of astrophysical and cosmological origin. The obstacles for the astrophysical description are firstly the lack of a concrete solution for the last parsec problem [59], secondly, the absence of large anisotropies in the NANOGrav signal [60] and thirdly the complete absence of isolated supermassive black hole merger events. Clearly, the detection of large anisotropies in the future may point out that the astrophysical description is plausible [61, 62]. However, some hint in this direction should be present even in the 2023 data, so the near future will shed light on these issues. On the other hand, there are a lot of cosmological scenarios that may explain the signal, like phase transitions, cosmic strings, primordial black holes and so on. With regard to the inflationary perspective, it seems that ordinary inflation cannot explain in any way the 2023 NANOGrav signal, unless a highly blue-tilted tensor spectral index is produced and a significantly low-reheating temperature is required [20, 21, 63, 64]. In principle, a blue-tilted tensor spectral index can be generated in the context of many theories, like string gas theories [65–67], or some Loop Quantum Cosmology scenarios can also predict such a tensor spectrum [68–71], and moreover the non-local version of Starobinsky inflation [72–74] which also yield an acceptable amount of non-Gaussianities. Furthermore, it is important to note that conformal field theory can yield a blue-tilted tensor spectrum [75]. From the aforementioned examples, the only which yields a strongly blue-tilted tensor spectral index is the non-local Starobinsky model, however in such a case a low reheating temperature is also needed. In the case that the reheating temperature is actually too low, the electroweak phase transition is put into peril, since it cannot be realized thermally, it is required that the reheating temperature is at least 100 GeV. We shall discuss this important issue at a later section. Thus the situation is somewhat perplexed since it is not certain what causes the 2023 NANOGrav signal. Only the combination of data coming from all the future experiments like LISA and the Einstein Telescope, including NANOGrav, may point out toward to a specific model behind the stochastic signal. This synergy between experiments may also help determining the reheating temperature.

In view of the above line of research, in this work we shall report an intriguing result in the context of inflationary theories and their ability to explain the 2023 NANOGrav signal. Our analysis requires a stiff, or nearly stiff pre-CMB era, which can be caused by various mechanisms, see for example [76–85]. The approach of Ref. [76] remarkably fits the line of research we shall adopt in this work, and the pre-CMB stiff or kination era is caused by the axion. Actually the

authors perform a thorough and scientifically sound analysis to put constraints on such a pre-CMB kination era caused by the axion. We shall not use a specific model in our work, however the constraints of [76] are taken into account in our analysis. Also it is noticeable that recently another work which invokes a stiff era and the possibility of explaining the NANOGrav 2023 signal appeared in the literature [86], in a different approach though, compared to the one we shall present in this work. Coming back to our case, the stiff era is required to correspond at a wavenumber range $k = 0.4 - 0.9 \text{ Mpc}^{-1}$, so just before the recombination era, and recall that the CMB pivot scale is at $k = 0.002 \text{ Mpc}^{-1}$. Note that the CMB modes with $k = 0.002 \text{ Mpc}^{-1}$ reentered the horizon at $z = 1100$ so during recombination, so the stiff era is assumed to have occurred before the recombination so when $k = [0.4, 0.9] \text{ Mpc}^{-1}$ hence for a redshift $z > 1100$. The inflationary era can be described by a standard red-tilted or a blue-tilted theory. With regard to the red-tilted inflationary era, it can be described by a standard inflationary scenario or some modified gravity [87–93], while the blue-tilted inflationary era can be described for example by an Einstein-Gauss-Bonnet gravity. The striking new result is that the tensor spectral index in the latter case is not required to be strongly blue-tilted, but it can be of the order $\sim \mathcal{O}(0.37)$, which can be easily generated by an Einstein-Gauss-Bonnet theory. In the case of a blue-tilted inflationary era, the compatibility with the NANOGrav result comes easily when the reheating temperature is of the order $T \sim \mathcal{O}(0.1) \text{ GeV}$, while the red-tilted inflationary case scenario is not detectable by the NANOGrav, however the prediction is that can be either detected by LISA or by the Einstein Telescope, not simultaneously though, depending on the reheating temperature and the total equation of state (EoS) parameter post-inflationary.

Regarding the stiff pre-CMB era, we shall assume an agnostic approach without proposing a model for this, except for the last section where we shall discuss a potentially interesting model.

The methodology of reconstruction that will be employed below is based on the work of [184]. Before we begin any sort of theoretical calculation it is important to note that we are working on the well known FLRW metric:

$$ds^2 = -dt^2 + a(t) \sum_{i=1,2,3} dx_i^2. \quad (2)$$

Now consider the action integral for an $f(R)$ gravity

$$S = \frac{1}{2k^2} \int f(R) \sqrt{-g} dx^4, \quad (3)$$

with k being the inverse of the Planck mass, as given by the relation $k = \frac{1}{M_{pl}}$. Now, varying the action with respect to the inverse metric tensor we obtain the equations,[143]

$$f_R(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f_R(R) = kT_{\mu\nu}. \quad (4)$$

The energy-momentum tensor is given by the equation:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \quad (5)$$

With L_m being the Lagrangian of matter, and the expressions f_R and f_{RR} represent the first, and second derivative of the function $f(R)$ with respect to the Ricci scalar R , respectively. The above relations when evaluated using the FLRW metric [2], and knowing that the Ricci scalar can be expressed as:

$$R = 6\dot{H} + 12H^2, \quad (6)$$

produce the following useful equation:

$$-\frac{f(R)}{2} + 3(H^2 + \dot{H})f_R - 18(4H^2\dot{H} + H\ddot{H})f_{RR} + \kappa^2\rho = 0 \quad (\text{Friedmann equation}), \quad (7)$$

which we can recognize as the Friedmann equation in the case of Einstein gravity [168]. To begin the reconstruction technique, we shall make the following steps to better accomplish constructing a differential equation in terms of R , $f(R)$ and its derivatives. We shall work in number of e-folds, defined as $N = \log(\frac{a}{a_0})$, with the current scale factor a_0 being normalized to unity. Also, we define the function $G(N) = H^2$. The Ricci scalar [184], thus becomes, $R = 3G'(N) + 12G(N)$, with G prime denoting the first derivative with respect to the number of e-folds, the G function can be written as:

$$G(N) = \frac{2n^2e^{-\frac{2N}{n}}(b^{1/n} - e^{N/n})^2}{b^2}, \quad (8)$$

using that fact we can solve the number for the number of e-folds in terms of the Ricci scalar. As the final expression is really cumbersome to work with, we shall make the large curvature approximation. The curvature at primordial times is orders of magnitude larger than the other parameters and thus, we can make an expansion of $N(R)$ around infinity. Keeping only the important terms we get the expression:

$$N = n \log \left(\frac{2 \left(\frac{-\sqrt{3}b^2 \sqrt{-nR} \sqrt{-2b^2nR+b^2R-3n}}{|24n^2-b^2R|} + \frac{24\sqrt{3}n^2 \sqrt{-n} \sqrt{-2b^2nR+b^2R-3n}}{|24n^2-b^2R|} + 12n^2 - 3n \right)}{24n^2 - b^2R} \right) + \log(b). \quad (9)$$

By, invoking a second cosmological equation, i.e the energy-conservation of the cosmological fluids, with a constant EoS parameter the total density can be written as:

$$\rho_{fluids} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N} \quad (10)$$

Substituting this equation in the Riemann equation we deduce the following differential equation:

$$9G(N(R))[4G'(N(R)) + G''(N(R))]F''(R) + (3G(N) + \frac{3}{2}G'(N(R)))F'(R) - \frac{1}{2}F(R) + \kappa^2 \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N(R)} = 0. \quad (11)$$

Our analysis, is focused on a field-free reconstruction process, and as such the part containing cosmic fluids will be omitted. Although it is of high interest to point out that this methodology, agrees with the results of [208]-[211], that approach the problem with the Raychaudhuri equation as the starting point of their analysis. Of course, our results coincide ,on account of the fact that the three equations that govern the dynamics of Cosmology (Friedmann, Raychaudhuri and continuity equation) are not linearly independent, and so by combining any two of them it is possible to derive the third one. In our case, the resulting differential equation, can be simplified if we assume the large curvature approximation, and keep only the most important polynomial terms that dominate during primordial times, as the rest of the expansion is in terms of inverse power of R . The differential equation and coefficients can be seen below:

$$A(R)F''(R) + B(R)F'(R) - \frac{1}{2}F(R) = 0, \quad (12)$$

with:

$$A(R) = \frac{R^2}{2n-1}, \quad B(R) = \frac{(n-1)R}{2(2n-1)}. \quad (13)$$

Solving the differential equation we obtain the form of the reconstructed gravity, that reproduces an emergent universe scenario, namely the $f(R)$ model is:

$$f(R) = c_1 R^{\frac{1}{4}(-\sqrt{n^2+10n+1}-n+3)} + c_2 R^{\frac{1}{4}(+\sqrt{n^2+10n+1}-n+3)}, \quad (14)$$

with c_1, c_2 being the constants of integration. One notable aspect of the above expression is that $f(R)$ gravities of the form $c_1 R^{m+} + c_2 * R^{m-}$ appear often in reconstruction procedures. Also in order to further support our argument of the large curvature approximation, we shall present the order of magnitude of the Ricci scalar during primordial times. The formula $R = 12H^2 + 6\dot{H} \approx 12H^2$. The authors of [188] find the scale of inflation to be $H \approx O(10^{21})eV$, therefore the Ricci scalar is $R \approx O(10^{42})eV$. It easily follows from the sufficiently large value of curvature (which is going to be orders of magnitude larger when the Universe is in its Einstein static phase), justifies the large curvature approximation as the higher order terms dwarf the rest of the expression in the series. Also, an inspection of the form of the $f(R)$ gravity reveals as stated in [202], the terms of the equations can be grouped as follows:

$$3*H^2 = \kappa^2 \rho_{tot}, \quad -2\dot{H} = \kappa^2(\rho_{tot} + P_{tot}), \quad (15)$$

with $\rho_{tot} = \rho_{radiation} + \rho_{gravity}$ and $P_{tot} = P_{radiation} + P_{gravity}$. The pressure and density due to gravity are purely geometric terms and are described by the equations:

$$\kappa^2 \rho_{gravity} = \frac{F_R R - F}{2} + 3H^2(1 - F_R) - 3H\dot{F}_R, \quad \kappa^2 P_{gravity} = \ddot{F}_R - H\dot{F}_R + 2\dot{H}(F_R - 1) - \kappa^2 \rho_{gravity} \quad (16)$$

It is then clear that, as the cosmological fluid is a perfect one, the gravitational pressure and density must follow the same continuity equation, namely:

$$\dot{\rho}_{gravity} + 3H(\rho_{gravity} + P_{gravity}) = 0, \quad (17)$$

and as such our $f(R)$ model is conserving.

II. PRIMORDIAL CURVATURE DYNAMICS

This section will be dedicated to deriving appropriate expressions for the parameters of that dictate the primordial profile for the scalar and tensor perturbations. The expression mentioned below are the same for inflationary mechanics, but due to the generality of their derivation they can be used to check the viability of our model. The methodology that will be presented below consists of writing easily manipulated expressions for the parameters of the emergent epoch, using them to calculate the spectral indices for scalar and tensor perturbations, as well as the tensor-to-scalar ratio, and lastly compare them to the results demonstrated by the Planck 2018 data ,[174]

The main assumptions of inflation are that $\ddot{H} \ll H\dot{H}$, and $\frac{\dot{H}}{H^2} \ll 1$ and the initial emergent epoch is described by the parameters:

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = 0, \quad \epsilon_3 = \frac{\dot{f}_R}{2Hf_R}, \quad \epsilon_4 = \frac{\ddot{f}_R}{H\dot{f}_R}. \quad (18)$$

Thus we get from [177],[178] ,the expressions for the observable quantities:

$$nS = 1 + \frac{-4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3}{1 - \epsilon_1}, \quad r = 48 \frac{\epsilon_3^2}{(1 + \epsilon_3)^2}, \quad nT = -2(\epsilon_1 + \epsilon_3). \quad (19)$$

By far the easier parameter to calculate is ϵ_1 , namely using the scale factor 1 we get:

$$\epsilon_1 = \frac{b^2 e^{\frac{2\sqrt{2}t}{b}} \left(e^{-\frac{\sqrt{2}t}{b}} + 1 \right)^2 \left(\frac{2ne^{-\frac{2\sqrt{2}t}{b}}}{b^2 \left(e^{-\frac{\sqrt{2}t}{b}} + 1 \right)^2} - \frac{2ne^{-\frac{\sqrt{2}t}{b}}}{b^2 \left(e^{-\frac{\sqrt{2}t}{b}} + 1 \right)} \right)}{2n^2}. \quad (20)$$

Now will try to express the other parameters as a function of ϵ_1 . From the known literature we can find that,

$$\epsilon_1 = -\epsilon_3(1 - \epsilon_4) \leftrightarrow \epsilon_3 = \frac{\epsilon_1}{\epsilon_4 - 1}. \quad (21)$$

The last parameter ϵ_4 is a little bit trickier to calculate, namely :

$$\epsilon_4 = \frac{\ddot{f}_R}{H\dot{f}_R} = \frac{f_{RRR}\dot{R}^2 + f_{RR}\dot{R}}{Hf_{RR}\dot{R}}. \quad (22)$$

But, for the time derivative of the Ricci scalar we get after assuming that $\ddot{H} \ll H\dot{H}$ (namely the first inflation condition):

$$\dot{R} = \frac{d}{dt}(12H^2 + 6\dot{H}) = 24\dot{H}H + 6\ddot{H} \approx 24\dot{H}H = -24H^3\epsilon_1. \quad (23)$$

Now calculating the time derivative for ϵ_1 ,

$$\dot{\epsilon}_1 = \frac{d}{dt} \frac{-\dot{H}}{H^2} = \frac{-\ddot{H}H^2 + 2H\dot{H}^2}{H^4} = \frac{-\ddot{H}}{H^2} + \frac{2\dot{H}^2}{H^3}. \quad (24)$$

After ignoring the second order derivatives of H , we approximately get:

$$\dot{\epsilon}_1 = \frac{2\dot{H}H}{H^4}. \quad (25)$$

Also setting the function $x(R) = -48\frac{f_{RRR}H^2}{f_{RR}}$, we get the final expression for the ϵ_4 :

$$\epsilon_4 = \frac{x(R)\epsilon_1}{2} - 3\epsilon_1 + 2\frac{\epsilon_1^2 H}{H\epsilon_1} = \epsilon_1 \left(\frac{x(R)}{2} - 1 \right). \quad (26)$$

The function $x(R)$, given in terms of the cosmic time, takes enough space in this letter to render it unreadable. For this reason , it could be computed either manually by the reader or seen at the attached Mathematica Notebook, provided in the documentation [187].

We wish to calculate the observational parameters, using the number of e-folds. To do that we integrate the expression $N = \int_{t_i}^{t_{early}} H(t) dt$, and thus we get the formula for the time of the first horizon crossing, in terms of the Number of e-folds(N). The time that the initial H conditions start to break down is when $\varepsilon_1 = 1$, that happens at time, $t_{final} = \frac{b \log(-n)}{\sqrt{2}}$. That is the reason n had to be negative, so that the final time is not imaginary. Now the integral is solved with a straightforward u-substitution, so that the time of exit in terms of the number of e-folds is:

$$t_i = \frac{b \log \left(\frac{1}{\frac{(n-1)e^{-\frac{N}{n}}}{n} - 1} \right)}{\sqrt{2}}. \quad (27)$$

The functions $n_S(t)$, $n_T(t)$ and $r(t)$, evaluated at the time of initial curvature perturbations(ti), contain 5 free parameters. The order of the parameters is [N,n,b,c₁,c₂]. Its obvious by now that the functions in question are extremely large, hundreds or thousands of characters in length and as such cannot be displayed in this letter. The process of finding the correct range of parameters was made extremely easy by utilizing the Genetic Algorithm showcased at [**Appendix**]. In general for $b > 1000$ the parameters don't present significant change, while being highly influenced by the value of n . Therefore for 50-60 e-folds, and b larger than 1000, the n value that makes the model viable is in the range of -30 to -60. An example is that the point [60, -60, 1000, 1, 1] produces $nS = 0.9630737$ and $r = 0.0042824$. These values comfort with the Planck 2018 constraints:

$$nS = 0.962514 \pm 0.00406408, \quad r < 0.064$$

It must be noted that the parameters c_1 and c_2 play no role in the viability of the model, as with the substitution mentioned earlier the ratio $\frac{f_{RRR}}{f_{RR}}$ found in ε_4 is not strongly dependant on the constants of integration c_1 , c_2 . Also the range of viability of the model naturally confirms the demand that b must be positive and, so that the scale does not dive into the negatives, while also having n smaller than zero for the universe to experience rapid development. The nT results remain of high interest as for some inputs of the free parameters the nT becomes negative and as such the results of our model will not be detected by the next generation of gravitational waves detectors.

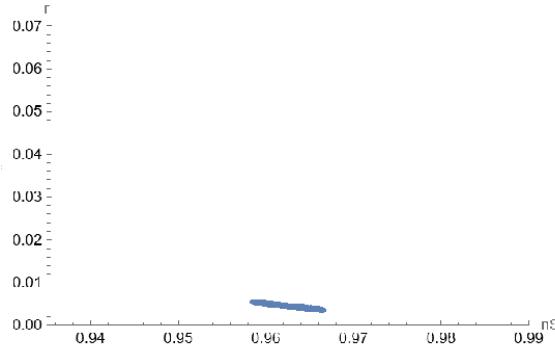


FIG. 2. The r-nS diagram for the viable points. It fits quite comfortably inside the range specified by Planck

The comoving Hubble Horizon represents the maximum distance information, travelling at the speed of light, can reach us. The mathematical expression is given by:

$$R_h(t) = \frac{1}{a(t) * H(t)}. \quad (28)$$

In our model, this takes the form:

$$R_h(t) = -\frac{e^{\frac{\sqrt{2}t}{b}} \left(e^{-\frac{\sqrt{2}t}{b}} + 1 \right)^{1-n}}{\sqrt{2n}}. \quad (29)$$

As presented in the plot below, the comoving Hubble radius plummets during the accelerated phase of our emergent universe scenario, a key characteristic that justifies treating that phase using inflationary dynamics. Now that we have been introduced to the indices ε_i , $i = 1, 3, 4$, there must be a discussion regarding the incompatibility of

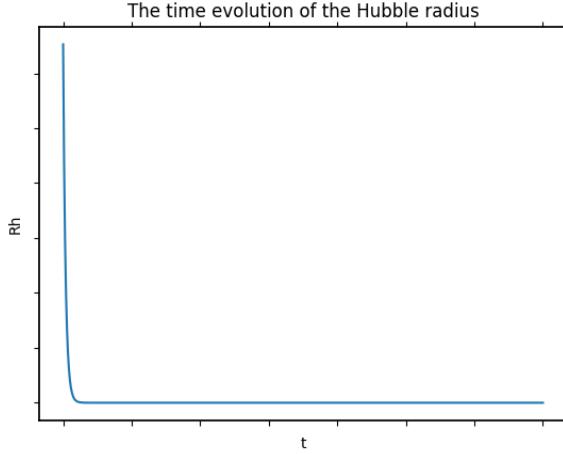


FIG. 3. The Hubble radius in terms of time, for $n=-60$ and $b=2000$

mainstream scale factors to the standard procedure of inflationary dynamics. Namely for scale factors of the form $a(t) = b(1 + e^t)^n$, fulfill the exit condition $\varepsilon_1 = 1$, when n is negative, but such a value for n contradicts the main idea of emergent universes for the scale factor plummets to zero instead of showing a rapid growth. Same problems appear when introducing more free parameters into the scale factor as the exit condition demands that the free variables make an unnatural scale factor inconsistent with the demands of an emergent universe. The same issues are presented in different scale factors such as: $a(t) = b * \ln(k + e^{t*p})$. These issues where the driving force behind the choice of the peculiar scale factor that we introduced at the start of the letter.

III. BIG BANG NUCLEOSYNTHESIS CONSTRAINTS

The following methodology relies on the work of [189],[190]. Having obtained the input parameters that fall in line with the Planck 2018 data concerning n_S and r , we shall examine the limitations being put in place by BBN data. First of all, the model we presented was not suitable for such an investigation, by adding a term of $+R$, we reach the new form of:

$$f(R) = R + c_1 R^{\frac{1}{4}(-\sqrt{n^2+10n+1}-n+3)} + c_2 R^{\frac{1}{4}(+\sqrt{n^2+10n+1}-n+3)}, \quad (30)$$

at first such a change may seem arbitrary, but there are valid grounds for such an addition. Specifically due to the form our differential equation 12, the residue of substituting the extra R term back in, is of order $0.1*R$. That deviation is absolutely acceptable if we consider the general large curvature approximation that was used to deduce the form of the $f(R)$ gravity. On the other hand, that term may not reproduce the scale factor introduced at the start of our letter, but a modified version of it, so that it still has the properties discussed. Another important thing to note is that our model for the viable n range takes the form $f(R) = c_1 * R^m + c_2 * R^k$, with $m \approx 2$ and $k \approx 20$, meaning it represents an enhanced Starobinsky model. As we will see later the c_2 constant is completely free meaning no matter what value it takes the viability of model is unchanged, but due to obvious physical arguments, at the end of our analysis we shall set equal to zero. Regarding the introduction of the R term it must be noted ,that because the ε_3 index depends only on the second and third derivative of the $f(R)$ gravity in terms of the Ricci scalar, it does not spoil our process of finding a viable range of parameters using the n_S and r parameters.

After exiting the inflationary phase, our Universe enters a radiation dominated era in which after the first few seconds the interaction between quarks freezes out and begins the process of making the nuclei of heavier elements and their isotopes, from protons and neutrons, in a process called Nucleosynthesis. Considering that due to adiabatic expansion the equality describing entropy $S = T^3 * a^3$ must not be violated, the temperature drops as $T \propto 1/a$. That small window of time in the order of a few minutes, and concerning energies of MeV to GeV, is really well understood as such energy scales have been studied and replicated in large particle colliders, resulting in some of the most precise observations on the field of Cosmology.

The Standard model of Cosmology (ΛCDM) , guided by the Friedmann equation in the scope of General Relativity,

describes the expansion of the universe in terms of time, during BBN, as:

$$H_{GR}^2 = \frac{M_P^{-2}}{3} \rho_r, \quad (31)$$

with ρ_r being the density of radiation that dominates during BBN. For relativistic species it is given in the form of :

$$\rho_r = \frac{\pi^2}{30} g_* T^4, \quad (32)$$

with $g_* \approx 10$ being the degrees of freedom of the relativistic species. Also it should emphasized that the constant M_P deviates from the Planck Mass, as $M_{PL} = \sqrt{8\pi} M_P = 1.22 * 10^{19}$ GeV

During BBN the total rate of conversion between protons to neutrons, due to interactions with the weak force, can be described by adding the rates of each of the subsequent reactions taking place. To be precise, the neutron conversion to proton reaction rate is:

$$\lambda_{pn}(T) = \lambda_{n+e^+ \rightarrow p+\bar{\nu}_e} + \lambda_{n+\nu_e \rightarrow p+e^-} + \lambda_{n \rightarrow p+e^- + \nu_e}, \quad (33)$$

along with the inverse reactions we get the total rate $\lambda_{total}(T) = \lambda_{pn} + \lambda_{np}$. Assuming that the energies are high enough for Boltzmann distribution to be able to describe the present processes, while the species are highly relativistic so that their masses are insignificant in comparison to their energies, we can derive the simplified expression:

$$\lambda_{total}(T) = 4AT^3(4!T^2 + 2 * 3!QT + 2!Q), \quad (34)$$

with Q being the difference between the masses of the two nucleons, and equal to $1.29 * 10^{-3}$ GeV, and A a constant equal to $1.02 * 10^{-11}$ GeV $^{-4}$. The Fermi-Dirac distribution given by: $p(T, \varepsilon_i) = \frac{1}{1 + \exp((\varepsilon_i - Q)/(k_B * T))}$. As $\varepsilon - Q \gg kT$ during this epoch, we can approximate it as $1 + \exp((\varepsilon_i - Q)/(k_B * T)) \approx \exp((\varepsilon_i - Q)/(k_B * T))$ leading back to Maxwell-Boltzmann statistics. As the Universe expands, and the reaction rate of this equation gets smaller and smaller as temperature drops, at some specific temperature T_{freeze} , the reaction freezes out. That happens when $\lambda_{total}(T) \approx 1/H$ leading to a freeze out temperature of:

$$T_f = \left(\frac{4\pi^3 g_*}{45 M_{PL}^2 c_q^2} \right)^{\frac{1}{6}} \approx 0.6 \text{ MeV}, \quad (35)$$

the constant c_q , being equal to $9.8 * 10^{-10}$ GeV $^{-4}$. Adding curvature terms to the standard GR model is going produce a different rate of expansion meaning the reactions will occur at different rates, leading to different mass ratios for the nuclei produced. Being more specific lets consider that our Hubble rate, after using equation 7 for modified gravities yields:

$$H^2 = \frac{M_P^{-2}}{3} (\rho_r + \rho_{grav}), \quad (36)$$

with the extra terms acting as the density of a fluid:

$$\begin{aligned} \rho_{grav} \left(\frac{M_P^{-2}}{3} \right)^{-1} &= \frac{1}{2} \left(-6^{\frac{1}{4}} (-\sqrt{n^2 + 10n + 1} - n + 3) \right) \\ &\times (H'(t) + 2H(t)^2)^{\frac{1}{4}(-\sqrt{n^2 + 10n + 1} - n + 3)} \\ &\times \left(c1 + c2 6^{\frac{1}{2}\sqrt{n^2 + 10n + 1}} (H'(t) + 2H(t)^2)^{\frac{1}{2}\sqrt{n^2 + 10n + 1}} \right) \\ &+ 3(H'(t) + H(t)^2) \\ &\times \left(\frac{1}{4} c2 \left(\sqrt{n^2 + 10n + 1} - n + 3 \right) (6H'(t) + 12H(t)^2)^{\frac{1}{4}(\sqrt{n^2 + 10n + 1} - n + 3) - 1} \right. \\ &\left. - \frac{c1 (\sqrt{n^2 + 10n + 1} + n - 3) (6H'(t) + 12H(t)^2)^{-\frac{1}{4}\sqrt{n^2 + 10n + 1} - \frac{n}{4}}}{4\sqrt[4]{6} \sqrt[4]{H'(t) + 2H(t)^2}} \right). \end{aligned} \quad (37)$$

Taking the square root of both sides and taking out the radiation density term we get the expression below, which is going to be just the GR Hubble rate along with a small deviation δH . As can be seen from the bibliography, this

is a common approach to tackle the gravitational effects during Nucleosynthesis. To make this point more concrete, let's consider the argument we used to justify the large curvature approximation. From the approximation $R \approx H^2$, while the universe undergoes a radiation dominated era after reheating, which dynamics will be left ambiguous besides the condition that it has short time-span, the Hubble rate scales as $H \sim t^{-1}$. Taking into consideration that after reheating the Hubble rate will be orders of magnitude smaller than the value of $O(21)$ eV, during inflation, and the many orders of magnitude that describe the timescale between the end of reheating and the start of BBN, we can safely conclude that $H \ll 1$. Finally, the higher curvature terms will be dominated by the simple Einstein-Hilbert part of gravity model, and will only act as perturbations. So, we can write the equation:

$$H = H_{GR} \sqrt{1 + \frac{\rho_{grav}}{\rho_r}} = H_{GR} + \delta H. \quad (38)$$

Using the above relationship, and taking into consideration that $\rho_r \gg \rho_{grav}$, as the gravitational effects are the first order perturbations:

$$\delta H = H_{GR} \left(\sqrt{1 + \frac{\rho_{grav}}{\rho_r}} - 1 \right) \delta H \approx H_{GR} \left(1 + \frac{\rho_{grav}}{2\rho_r} / - 1 \right) = H_{GR} \frac{\rho_{grav}}{2\rho_r} \quad (39)$$

As during BBN, the Hubble rate and the freeze-out temperature are, in leading order, related by:

$$H = \lambda_{tot} \sim qT^5 \quad (40)$$

Therefore, a deviation from the H_{GR} will produce an according deviation for the freeze-out temperature, T_f . These deviations, after combining the previous equations give the final expression:

$$\frac{\delta T_f}{T_f} = \frac{\rho_{grav}}{\rho_r} \frac{H_{GR}}{10c_q T_f^5}. \quad (41)$$

The above equation is well determined as it is crucial for the calculation of the mass fraction of ${}^4\text{He}$. The observational results determine that it must be:

$$\frac{\delta T_f}{T_f} < 4.7 * 10^{-4}. \quad (42)$$

Applying the presented methodology to our model, and demanding that the inequality must be respected we get a really satisfactory result. The c_2 parameter is irrelevant to our analysis, and it can take any value without spoiling the BBN results while the c_1 parameter is restricted by a constant P of $O(128 - 129)$ depending on the choice of the value of n . That means that for any value of $-P < c_1 < P$, our model does not produce any changes to the results of BBN, thus omitting the need for fine tuning the integration parameters and presenting an excellent opportunity for a viable $f(R)$ model even during late times.

CONCLUDING REMARKS AND DISCUSSION

The present letter, after taking into consideration the inconsistencies at producing imaginary exit times and unnatural models for an emergent universe scenario, explored the viability of scale factor 1. We used the techniques developed at [184], and reconstructed the form of $f(R)$ in the matter-free, flat universe approach that reproduces that specific scale factor at large curvatures. Using the inflationary parameters $\varepsilon_i, i = 1, 2, 3$, it was possible to extract the spectral indices for scalar and tensor perturbations along with the tensor-to-scalar ratio, and find the appropriate range for the parameters that satisfies the constraints set by Planck 2018 data, for 50-60 e-folds. The parameter b should be at least of order $O(3)$, with the appropriate choice of n being between -30 to -60, while also noting that the free constants of integration played no role in the whole analysis. Going even further, we introduced an extra linear term, $+R$ that is just the normal expression for General Relativity, and explored the ability of the model to reproduce BBN constraints. After carefully going throughout the process presented at III, we found the constant c_2 has no significance on the results of Nucleosynthesis while the constant c_1 can take virtually any value, negative or positive, up to order of $O(128)$. The model has the form of an enhanced Starobinsky model, while the problematic term of large curvature can easily be extinguished without any issues to our model. Therefore the present $f(R)$ is able to reproduce the inflationary period, the BBN along with an emergent universe scenario that bypasses the quantum gravity and singularity regime.

ACKNOWLEDGMENTS

This research has been funded by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP19674478) (V.K. Oikonomou). G. Kafanelis is indebted to E.N. Saridakis for his guidance on implementing the BBN constraints, and also to his fellow colleague Mr Fotis Fronimos for his ingenious insights and productive conversations.

APPENDIX

In this section we shall present the genetic algorithm that enables us to find the viable points efficiently and reliably. The class GenAlgo represents the most vital part of the code base. During initiation the constructor expects the n_S and r functions, that were calculated in the framework of the working theory, the symbols used in them in the form of a **Sympy.symbols** object, and lastly a string that represents the fitness function that will be used during the genetic algorithm. We leverage the speed due to vectorization that is provided by the **Sympy.lambdify** object to speed up the numerical calculations. As our group mostly works with Mathematica Notebooks, the object allows raw Mathematica format to be transformed into appropriate **Sympy** expressions.

```
class GenAlgo:
    def __init__(self,str_nS,str_r,input_symbols,mathematica_form=True,
                 fitness_expression='devr * devr + devnS * 100 + devN * 0.1'):
        if mathematica_form:
            self.python_expr_r = parse_mathematica(str_r)
            self.python_expr_nS = parse_mathematica(str_nS)
        else:
            self.python_expr_r = str_r
            self.python_expr_nS = str_nS
        self.function_symbols = input_symbols
        self.fitness_expression = fitness_expression

        #The Planck 2018 constaints
        self.rlim = 0.064
        self.nSmin = 0.962514 - 0.00406408
        self.nSmax = 0.962514 + 0.00406408

        #The range of the e-folds
        self.Nmax = 60
        self.Nmin = 50

        self.nSsympyFunction = sym.lambdify(self.function_symbols,self.python_expr_nS,'numpy')
        self.rsympyFunction = sym.lambdify(self.function_symbols,self.python_expr_r,'numpy')

        self.fitness_expression = fitness_expression
        self.scoringSymbols = sym.symbols('devnS devr devN')
        self.scoringFunction = sym.lambdify(self.scoringSymbols,self.fitness_expression,'numpy')
        pass
```

Each symbol in these expressions represents a parameter of our model, so for the n_S and r functions to be able to take the appropriate inputs and produce a numerical result, they are defined in the form presented below. We used the symbols **Ne1 n b c1 c2** to represent the free parameters in the model of 1, always starting with the number of efolds, but they can be changed to fit other phenomenological models with different parameters. It must be noted that if the evaluated expression is imaginary, **Sympy** will throw an **Exception** and thus the functions will return a value of positive infinity, the reason behind this choice will be explained thoroughly when we introduce the fitness function.

```
def nS(self,args):
    try:
        with warnings.catch_warnings():
            warnings.filterwarnings("error")
```

```

        var = self.nSsympyFunction(*args)
        return var
    except Exception as e:
        return float('inf')

def r(self,args):
    try:
        with warnings.catch_warnings():
            warnings.filterwarnings("error")
            var = self.rsympyFunction(*args)
            return var
    except Exception as e:
        return float('inf')

```

We shall break off our explanation of the **GenAlgo** to introduce another integral component of our code-base. The **Solution** object holds a **NumPy** array, which contains the parameter values in the same order as they were introduced in the n_S and r functions. Additionally, the **Solution** object incorporates a crucial function called **score**, serving as the fitness function within our algorithm. The **scoringFunction** is passed from the **GenAlgo** during the computation process. The **dev_keyword** calculates the absolute difference of the 'keyword' from its appropriate bounds. The bounds for each 'keyword' is transferred from the **GenAlgo** object. Along with satisfying the Planck 2018 constraints we impose the condition that the number of e-folds must be in the 50-60 range to appropriate address the Homogeneity problem. In the scoring function, we minimally weight the deviation from the constraint in the r function to diminish its impact on the overall score. During training, when all deviations were considered equally, the algorithm tended to excessively prioritize reducing the deviation from the r constraint since there is no lower bound, thus resulting in a skewed optimization. Furthermore, we carefully select the weights so that the deviations in both n_S and N have comparable magnitudes. It's important to note that, contrary to the typical evaluation of fitness algorithms, our algorithm is rewarded for achieving lower scores, indicating a higher likelihood of generating a viable phenomenology within the current values stored in the **Solution** object. In order to address this confusion, we named the function "score" instead of the standard "fitness". One last comment concerning the **score** function is that, as was mentioned earlier the n_S and r functions return positive infinity when the evaluated expression has an imaginary part, consequently **Solutions** that produce non real results are quickly ruled out by the algorithm.

```

class Solution:
    def __init__(self, genome):
        self.genome = genome

    def score(self,other):
        nS_value = other.nS(self.genome)
        r_value = other.r(self.genome)
        efolds = self.genome[0]
        dev_nS = abs(nS_value-other.nSmin)+abs(nS_value-other.nSmax)
        dev_r = abs(r_value-other.rlim)
        dev_N = abs(efolds-other.Nmin)+abs(efolds-other.Nmax)
        objectScore = other.scoringFunction(dev_nS,dev_r,dev_N)
        return objectScore

```

To initiate the algorithm, the function **genetic_algo_start** is invoked. This function accepts several input parameters, including the range for each symbol, the desired population size of **Solutions** for training, the quantity of the top-performing individuals in each generation that will participate in the crossover for the subsequent generation, and the total number of iterations for which the algorithm will execute. Additionally, if a pre-existing trained population is available, it can be employed as the initial population.. Each population consists of an array containing **Solutions**.

```

def start(self,bounds, size_population, fit_lim, iterations, loaded_population=None):
    #Making the first population manually
    fit_lim = fit_lim+1
    best_scores=np.zeros((iterations,fit_lim+1),dtype=float)
    pops = np.zeros(iterations+1, dtype='object')
    best_pops = np.zeros(iterations+1, dtype='object')
    population = np.empty(size_population, dtype='object')

```

```

if loaded_population is None:
    for i in range(size_population):
        var_genome_0 = self.generator(bounds)
        sol = Solution(genome=var_genome_0)
        population[i] = sol
elif isinstance(loaded_population, np.ndarray) and loaded_population.dtype == object:
    population = loaded_population[-1]
if len(bounds) != len(self.function_symbols):
    raise ValueError(f"Different dimensions.
        Bounds dimensions : {len(bounds)} and total symbols : {self.function_symbols}")

```

The initial population is constructed using the **generator** method of the **GenAlgo** class. This function takes the relevant bounds, which determine the range for each variable, and generates an array of the same length. Each element in the array corresponds to a random value within the specified bounds for the corresponding variable. To address a potential issue with the **random.uniform** command, which tends to under represent smaller numbers when the bounds differ significantly (up to three orders of magnitude), we employ a scaling technique. Firstly, we apply a logarithmic scaling using \log_{10} to reduce the bounds. Then, we generate the random input within this scaled range. Finally, we scale the generated value back up while preserving the appropriate sign, and append it to the final array. This approach ensures a more balanced representation of all possible values for a specific variable, even when there are substantial discrepancies between the upper and lower bounds. However, it is worth noting that this implementation requires non-zero bounds. Nevertheless, this limitation can be easily resolved by choosing a suitably small bound value.

```

def generator(self,b):
    n = len(b)
    output = np.zeros(n)
    for index, value in enumerate(b):
        lower_bound,upper_bound = value
        l_b, u_b = (math.log10(abs(lower_bound)) , math.log10(abs(upper_bound)) )
        cons_sign = -1 if lower_bound < 0 or upper_bound < 0 else 1
        output[index] = (10**((random.uniform(l_b,u_b))))*cons_sign
    return output

```

Next, begins the process of calculating the score for each player in the current generation. The scores of the **Solutions** are stored in the **scoreboard** array. By sorting the **scoreboard** array we find the indices of the **fit_lim** lowered scored players, and create a new array that contains those solutions, called **best_candidates**, and append it to the larger array **best_pops**

```

for generation in range(iterations):
    #Initializing
    t1 = time.time()
    scoreboard = np.zeros(size_population)
    population = pops[generation]

    #Scoring
    scoreboard = parallel(population)

    best_candidates_pos = np.argpartition(scoreboard, fit_lim+1)[:fit_lim+1]
    #Fitness
    best_candidates = np.zeros(fit_lim+1,dtype='object')
    index_position = 0
    for position in best_candidates_pos:
        var = population[position]
        best_candidates[index_position] = var
        index_position = index_position +1
    best_pops[generation]=best_candidates.copy()

```

During the subsequent phase of the algorithm, we evenly distribute the **genome** (the array of values) of each of the top-performing players into a temporary array called **genetic_tree**. To illustrate this, let's consider an example where our population consists of 1000 elements, and we choose to retain the 5 best players. In this scenario, the first

200 elements of the **genetic_tree** array would correspond to the genome of player 0, the next 200 elements would correspond to the genome of player 1, and so on. By doing this, we ensure that all players have an equal representation in the **genetic_tree**, placing them on a level playing field for the next steps of the algorithm.

```
#Crossover
for indicator in range(len(ranges)-1):
    lower = math.floor(ranges[indicator])
    upper = math.floor(ranges[indicator+1] )
    var_genome = best_candidates[indicator].genome
    for index in range(int(lower),int(upper),1):
        genetic_tree[index]=var_genome
```

On to the stage of mutation. The **mutate** function, takes as input the **gene** of a **genome**, adds a random value between $(-\text{gene}/s, \text{gene}/s)$ and returns the modified **gene**. The value of s is set to 1, as we found through experimentation that it leads to a quicker and more stable convergence rate.

```
def mutate(self,gene): # The mutating function
    self.scale = 1
    return gene + np.random.uniform(-gene/self.scale,gene/self.scale)
```

After creating a **new_population** that contains the mutated **genomes** of the **best_candidates**, we replace the first elements of the array the **Solutions** with **best_candidates**. The reason behind this execution is that due to pure randomness the next generation can score higher than the current generation, as the **genomes** can mutate away from the local optimum. By incorporating the **best_candidates** into **new_population**, we guarantee the best players of the next generation score as low as the best players of the current one. Lastly we append **new_generation** into **pops**.

```
#Mutate
new_population = np.array([Solution(np.array([mutate(gene) for gene in tree])) for tree in genetic_tree])
#Keeping the best players of the previous generation
for i in range(len(best_candidates)):
    new_population[i]=best_candidates[i]

pops[generation+1] = new_population
```

The function returns the arrays **best_pops** and **pops**. In the notebook found at, [187], we use the **clear** and **get_values** functions, that in conjunction return the truly viable ranges found by the algorithm along with the values of the n_S and r functions that they produce.

```
def process_point(self,obj):
    genome = obj.genome
    test_nS = self.nS(genome)
    test_r = self.r(genome)
    if self.nSmin < test_nS < self.nSmax and test_r < self.rlim:
        return [test_nS, test_r]
    else:
        return [None,None]

def clear_points(self,arr):
    flattened = np.concatenate(arr).ravel()
    values = []
    nS_arr = []
    r_arr = []
    for obj in flattened:
        returnValue = self.process_point(obj)
        if returnValue[0] is not None:
            values.append(obj.genome)
            nS_arr.append(returnValue[0])
            r_arr.append(returnValue[1])
    return values, nS_arr, r_arr
```

To speed up the numerical calculations and bypass the GIL, the **RunParallel** function can be invoked that runs many populations in parallel using the **Pathos** module for a single **GenAlgo** object. It maps the provided list of arguments to the **ParallelStart** method in the **GenAlgo** object. Below the specifications for each function are presented. The **RunParallel** function utilizes a different way of clearing the points, and as such returns the total score accounting for all the parallel populations, along with all the calculated viable points, and their n_S , r values.

```
def RunParallel(obj,argsList):
    numCPUS = pathos.helpers.cpu_count()
    pool = ProcessingPool(nodes=numCPUS)

    args_list = argsList

    resultsParallel = pool.map(obj.ParallelStart, args_list)

def ParallelStart(self,args):
    bounds, size_population, fit_lim, iterations = args
    loaded_population = None
```

By running the algorithm with the appropriate forms of our n_S and r functions, that where extracted from the Mathematica Notebook, with the **RunParallel** function, and the following parameters

```
input_symbols = sym.symbols('Ne1 b n c1 c2')
fit_expr = 'devr * 0.1 + devnS * 50 + devN * 0.1
set_bounds = np.array([(50,60),(-10**20),-10**(-20)),
(-10**20),+10**(-20)),(-10**20),10**20),(-10**20),10**20)])
args_list = [(set_bounds,5000,20,15)]*16
```

The algorithm after running for about 1 minute , managed to calculate a total of 1.2 million points and produce the following diagram for the values of n_S and r , with 5086 viable points.

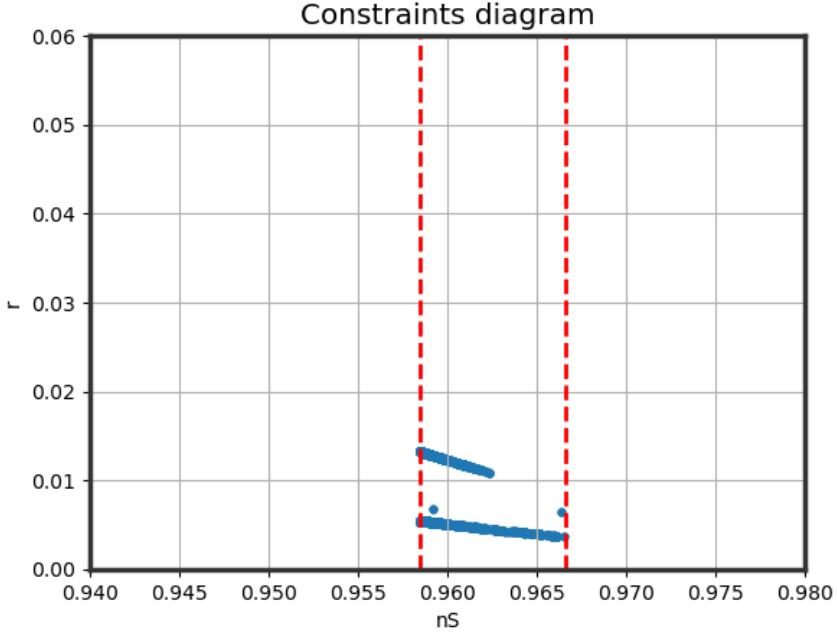


FIG. 4. n_S - r diagram for the viable points found by the algorithm

As it is evident, the algorithm is extremely fast and efficient at finding the optimum set of parameters to fit our constraints, all while being extremely user and developer friendly. We hope the algorithm is used in the work of other inflation researchers as it automates a rather tedious and time consuming task. Any feedback or suggestions for improving the algorithm are more than welcome, and we can be contacted through the emails specified at the

start of the letter.

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