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Integrable surfaces induced by generalized Landau-Lifshitz equation with self-consistent potential

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Abstract. In this paper, we present integrable surfaces associated with generalized Landau-Lifshitz equation with self-consistent potential. We obtained the first and second fundamental forms. The first fundamental form allow us to calculate the curvature and metric properties of a surface, particularly, length and area of related space. The second fundamental form determines the external geometry of the surface in the vicinity of this point. Together they permit to define extrinsic invariants of the surface and its principal curvatures. The results can be used to describe spin waves in magnets and ferromagnets.

1. Introduction

The growing interest in the physics of magnetic phenomena is due to the difference in the interaction between the local magnetic moments of ions. They can diverse in their nature and have a wide range of energy scales; also, due to the characteristics of the crystal structure, they differ greatly between magnetic ions neighboring in different directions. The anisotropy property can lead to the formation of ferro- or antiferromagnetic states, as well as unusual magnetic structures spin waves or magnons.

In this work, we consider generalized Landau-Lifshitz equation (GLLE) with self-consistent potential. GLLE is a well-known equation describing magnetization waves in magnets, or simply, spin waves [1]-[7]. We present integrable surfaces associated with GLLE with self-consistent potential. The first and the second fundamental forms were obtained. The first fundamental form is used to calculate the curvature and metric properties of a surface, particularly, length and area of related space. The second fundamental form determines the external geometry of the surface in the vicinity of this point. Together they permit to define extrinsic invariants of the surface and its principal curvatures. The results can be used to describe spin waves in magnets and ferromagnets.

2. Landau-Lifshitz equation with self-consistent potential

The generalized Landau-Lifshitz equation with self-consistent potential is one of the generalizations of Landau-Lifshitz equation. It reads as

$$\mathbf{S}_t + \frac{1}{2}\mathbf{S} \wedge \mathbf{S}_{xx} + \frac{2}{a}\mathbf{S} \wedge \mathbf{W} \quad = \quad 0, \tag{1}$$

$$\mathbf{W}_x + 2a\mathbf{S} \wedge \mathbf{W} = 0, \tag{2}$$

where $\wedge$ denotes a vector product, $\mathbf{S} = (S_1, S_2, S_3)$, $\mathbf{W} = (W_1, W_2, W_3)$ are vectors with lengths $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$, $\mathbf{W}^2 = W_1^2 + W_2^2 + W_3^2 = b$ and a, b are const. Also it would be important to note that in case of $W = 0$ system of equations (1)-(2) reduced to Classical Heisenberg Model. In most cases, it is convenient to work with the matrix form of this equation, which has the form

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{a}[S, W] = 0, \tag{3}$$

$$iW_x + a[S, W] = 0, \tag{4}$$

where $a = const$, $S = \Sigma_{j=1}^{3} S_j(x, y, t)\sigma_j$ is a matrix analogue of the spin vector, W - potential with the matrix form $W = \Sigma_{j=1}^{3} W_j(x, y, t)\sigma_j$,

$$S(x,t) = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad W(x,t) = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}, \tag{5}$$

and σ_j are Pauli matrices. The GLLE with self-consistent potential is gauge equivalent to Schrödinger-Maxwell-Bloch equation and corresponding investigations were made in works [8]-[11].

In mathematics, in the theory of integrable systems, a Lax pair is a pair of time-dependent matrices or operators that satisfy a corresponding differential equation are called the Lax equation. Lax pairs were introduced by Peter Lax to discuss solitons in continuous media [12]. The inverse scattering transform makes use of the Lax equations to solve such systems.

The LLE equation with self-consistent potential is integrable and its Lax representation can be written in the form

$$\Phi_x = U\Phi, \tag{6}$$

$$\Phi_t = V\Phi, \tag{7}$$

where the matrix operators U and V have the form

$$U = -i\lambda S, \tag{8}$$

$$V = \lambda^2 V_2 + \lambda V_1 + \left(\frac{i}{\lambda + a} - \frac{i}{a}\right) W. \tag{9}$$

Here

$$V_2 = -2iS, \quad V_1 = SS_x. \tag{10}$$

The compatibility condition of equations (6)-(7) give us

$$U_t - V_x + [U, V] = 0, \tag{11}$$

which is the zero curvature condition. Substituting U and V operators (8)-(10) in equation (11) we obtain the GLLE with self-consistent potential in matrix form (3)-(4).

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3. The fundamental forms for the GLLE with self-consistent potential

In general, the first and second fundamental forms (FF) are

$$I = g_{ij} dx^i dx^j, \tag{12}$$

$$II = b_{ij} dx^i dx^j, \tag{13}$$

here g_{ij}, b_{ij} are matrices

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tag{14}$$

$$b_{ij} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \tag{15}$$

The FF can be presented through position and normal vectors:

$$I = \mathbf{dr} \cdot \mathbf{dr} = \mathbf{r}_x^2 dx^2 + 2\mathbf{r}_x \mathbf{r}_t dxdt + \mathbf{r}_t^2 dt^2, \tag{16}$$

or

$$I = Edx^2 + 2Fdxdt + Gdt^2 \tag{17}$$

and

$$II = -\mathbf{dn} \cdot \mathbf{dr} = (\mathbf{n} \cdot \mathbf{r}_{xx})\, dx^2 + 2\,(\mathbf{n} \cdot \mathbf{r}_{xt})\, dxdt + (\mathbf{n} \cdot \mathbf{r}_{tt})\, dt^2 \tag{18}$$

or

$$II = Ldx^2 + 2Mdxdt + Ndt^2. \tag{19}$$

The position vector is

$$\mathbf{r} = (r_1, r_2, r_3) \tag{20}$$

and normal to the surface is

$$\mathbf{n} = (n_1, n_2, n_3), \quad \mathbf{n}^2 = 1. \tag{21}$$

Using Sym-Tafel formula [13]

$$r = \Phi^{-1} \Phi_\lambda, \tag{22}$$

we can obtain this vectors in matrix forms

$$r_x = \Phi^{-1} U_\lambda \Phi, \quad r_t = \Phi^{-1} V_\lambda \Phi. \tag{23}$$

Relations between derivations of vector and matrix form of r with respect to x and t:

$$\mathbf{r}_x^2 = \frac{1}{2} tr\left(r_x^2\right), \tag{24}$$

$$\mathbf{r}_t^2 = \frac{1}{2} tr\left(r_t^2\right), \tag{25}$$

$$\mathbf{r}_x \mathbf{r}_t = \frac{1}{2} tr\left(r_x r_t\right). \tag{26}$$

Now, we obtain the necessary quantities

$$r_x^2 = \Phi^{-1} U_\lambda^2 \Phi, \tag{27}$$

$$r_t^2 = \Phi^{-1} V_\lambda^2 \Phi, \tag{28}$$

$$r_x r_t = \Phi^{-1} U_\lambda V_\lambda \Phi. \tag{29}$$

The first fundamental form for the GLLE with self-consistent potential can be obtained by collecting and putting into (17) and necessary coefficients

$$\begin{aligned} I &= dx^2 + \left(8\lambda - \frac{1}{(\lambda+a)^2}Tr\{S,W\}\right)dxdt + \left(16\lambda^2 - \frac{1}{2}Tr((SS_x)^2)\right)dt^2 \\ &+ \left(\frac{2\lambda}{(\lambda+a)^2}Tr\{S,W\} + \frac{i\lambda}{2(\lambda+a)^2}Tr\{SS_x,W\} + \frac{1}{2(\lambda+a)^2}Tr(W^2)\right)dt^2, \end{aligned} \tag{30}$$

where $\{S,W\} = SW + WS$ - anticommutator. When $\lambda = \lambda_0 = 0$ then we can rewrite the first fundamental form as

$$\begin{aligned} I &= dx^2 - \frac{1}{a^2}Tr(SW)dxdt + \\ &+ \frac{1}{2}\left(Tr((SS_x)^2) - \frac{i}{a^2}Tr\{SS_x,W\} - \frac{4}{a^2}Tr(W^2)\right)dt^2, \end{aligned} \tag{31}$$

To obtain the second fundamental form, let us find the following

$$r_{xx} = \Phi^{-1}[U_\lambda, U]\Phi, \tag{32}$$

$$r_{xt} = \Phi^{-1}[U_\lambda, V]\Phi, \tag{33}$$

$$r_{tt} = \Phi^{-1}[V_\lambda, V]\Phi. \tag{34}$$

And a normal to surface can be calculated by

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_t}{|\mathbf{r}_x \wedge \mathbf{r}_t|}, \tag{35}$$

or

$$n = \frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2}tr\left([U_\lambda, V_\lambda]^2\right)}}. \tag{36}$$

Traces are determined as

$$tr(n \cdot r_{xx}) = \frac{tr([U_\lambda, V_\lambda][U_\lambda, U])}{\sqrt{\frac{1}{2}tr\left([U_\lambda, V_\lambda]^2\right)}}, \tag{37}$$

$$tr(n \cdot r_{xt}) = \frac{tr([U_\lambda, V_\lambda][U_\lambda, V])}{\sqrt{\frac{1}{2}tr\left([U_\lambda, V_\lambda]^2\right)}}, \tag{38}$$

$$tr(n \cdot r_{tt}) = \frac{tr([U_\lambda, V_\lambda][V_\lambda, V])}{\sqrt{\frac{1}{2}tr\left([U_\lambda, V_\lambda]^2\right)}}. \tag{39}$$

Here we came to interesting results normal vector to surface is equal to zero, cause trace of commutator $Tr[U_\lambda, V_\lambda] = 0$. Consequently, construction of the second fundamental form for considered model by this approach is impossible. It means that GLLE with self-consistent potential associated only with trivial soliton surface which proved by the existence of the first fundamental form only. For some equations soliton surfaces does not the same as integrable surface.

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4. Integrable surfaces induced by GLLE with self-consistent potential

In this section we want to present integrable surfase induced by GLLE with self-consistent potential. In a purpose of this lets make the following transformation $\mathbf{r}_x \equiv \mathbf{S}$. Then we can rewrite $\mathbf{r}_t$ as

$$\mathbf{r}_t = \frac{i}{2}\mathbf{r}_x \wedge \mathbf{r}_{xx} + \frac{1}{a^2}\mathbf{W}. \tag{40}$$

In this case, the first fundamental form denotes as

$$\begin{aligned} I &= dx^2 + \frac{2}{a^2}\mathbf{S}\mathbf{W}dxdt + \\ &+ \left(\frac{1}{4}(\mathbf{S}\wedge\mathbf{S}\mathbf{S}_x)^2 - \frac{1}{a^2}(\mathbf{S}\wedge\mathbf{S}_x)\mathbf{W} + \frac{1}{a^4}\mathbf{W}^2\right)dt^2, \end{aligned} \tag{41}$$

Now using (35) we can define normal to surface

$$\mathbf{n} = \frac{\frac{1}{2}\mathbf{r}_{xx} - \frac{1}{a^3}\mathbf{W}_x}{|\frac{1}{2}\mathbf{r}_{xx} - \frac{1}{a^3}\mathbf{W}_x|}, \tag{42}$$

or

$$\mathbf{n} = \frac{\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}. \tag{43}$$

Then determine coefficients of the 2^{nd} fundamental form

$$L = \mathbf{n}\cdot\mathbf{r}_{xx}, \tag{44}$$

$$M = \mathbf{n}\cdot\left(\frac{1}{2}(\mathbf{r}_x\wedge\mathbf{r}_{xx})_x + \frac{1}{a^2}\mathbf{W}_x\right), \tag{45}$$

$$N = \mathbf{n}\cdot\left(\frac{1}{2}(\mathbf{r}_x\wedge\mathbf{r}_{xx})_t\right), \tag{46}$$

or in term of $\mathbf{S}$ will look like

$$L = \frac{(\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x)\cdot\mathbf{S}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}, \tag{47}$$

$$M = \frac{\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}\cdot\left(\frac{1}{a^2}\mathbf{W}_x - \frac{1}{2}(\mathbf{S}\wedge\mathbf{S}_x)_x\right), \tag{48}$$

$$N = -\frac{1}{2}\frac{\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}\cdot(\mathbf{S}\wedge\mathbf{S}_x)_t. \tag{49}$$

The second fundamental form can be written as

$$\begin{aligned} II &= \frac{(\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x)\cdot\mathbf{S}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}dx^2 + \\ &2\ \frac{\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}\cdot\left(\frac{1}{a^2}\mathbf{W}_x - \frac{1}{2}(\mathbf{S}\wedge\mathbf{S}_x)_x\right)dxdt + \\ &- \frac{1}{2}\frac{\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x}{|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|}\cdot(\mathbf{S}\wedge\mathbf{S}_x)_t dt^2. \end{aligned} \tag{50}$$

5. Area of surface for the GLLE with self-consistent potential

Surface's area is given in the form

$$S = \int \sqrt{g} dxdt = \int \int |\mathbf{r}_x \wedge \mathbf{r}_t| \, dxdt. \tag{51}$$

Then our required surface area S will look like

$$S = \int \left| \frac{i}{a^3} \mathbf{W} - \frac{1}{2} \mathbf{r}_x \right| dt, \tag{52}$$

or

$$S = \int \left| \frac{i}{a^3} \mathbf{W} - \frac{1}{2} \mathbf{S} \right| dt. \tag{53}$$

The Gaussian curvature of the surface is given by

$$K = \frac{LN - M^2}{EG - F^2}. \tag{54}$$

Which is for our model will be

$$K = -\frac{\left[\frac{1}{2}(\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x) \cdot \mathbf{S}_x\right] \left[\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x \cdot (\mathbf{S} \wedge \mathbf{S}_x)_t\right] + \left[\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x \cdot \left(\frac{1}{a^2}\mathbf{W}_x - \frac{1}{2}(\mathbf{S} \wedge \mathbf{S}_x)_x\right)\right]^2}{\left[|\mathbf{S}_x - \frac{1}{a^3}\mathbf{W}_x|^2\right] \left[\frac{1}{4}(\mathbf{S} \wedge \mathbf{S}\mathbf{S}_x)^2 - \frac{1}{a^2}(\mathbf{S} \wedge \mathbf{S}_x)\mathbf{W} + \frac{1}{a^4}\mathbf{W}^2 - \left(\frac{1}{a^2}\mathbf{S}\mathbf{W}\right)^2\right]}, \tag{55}$$

and finally we found Gaussian curvature in terms of $\mathbf{S}$.

6. CONCLUSIONS

In this paper, using the Sym-Tafel formula, we obtained the first fundamental form of GLLE with self-consistent potential. Proved that for considered equation soliton surface does not equal to integrable surface induced by them. Then to construct integrable surface induced by GLLE with self-consistent potential we found the first and the second fundamental forms,area of surface and a Gaussian curvature. The results can be used to describe spin waves in magnets and ferromagnets. According to the results of this work, research in this direction is entering a new step, which will allow a more detailed study of the construction of different integrable surfaces associated with soliton equations.

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