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# Teleparallel gravity with non-minimally coupled $f$ -essence via Noether symmetry approach

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**Abstract.** In this paper, we examine teleparallel gravity with non-minimally coupled with non-canonical fermionic fields ( $f$ -essence). Noether symmetry approach can be used to fix the forms of coupling  $F(\Psi)$  and the potential  $V(\Psi)$  functions of the fermionic fields. In the context of the Friedman-Robertson-Walker metric, we investigate cosmological solutions of the field equations using these forms obtained by the existent of Noether symmetry.

## 1. Introduction

Astrophysical observational dates of recent years have shown that our Universe is expanding with an accelerated phase [1,2]. Various cosmological scenarios have been proposed to explain this interesting behavior. Many recent works admits the existence of a component of the Universe the so-called dark energy which is the responsible for the present accelerated expansion of the Universe, but whose nature still remains unknown. The cosmological constant was originally the most natural candidate for dark energy, but other dark energy cosmological models have also been proposed. The most common cosmological models, also called quintessence models, make use of scalar fields and barotropic equations of state where the ratio between the energy density and pressure assumes negative values. Other cosmological models consider is described by exotic equations of state like the Chaplygin gas or the van der Waals fluid.

Einstein's general relativity (GR) is well studied and tested in many works, alternative theories of gravity continue to be of considerable interest. One kind of these alternative theories of gravity assumes that the motion in the gravitational field is no longer geometrized, as in GR, but is encoded in a dynamic gravitational force, as in teleparallel gravity [3,4]. Teleparallel gravity may consider as a gauge theory for the translation group. As such, its fundamental field is neither the metric nor the tetrad, but a gauge potential assuming values in the Lie algebra of the translation group. This gauge character makes of teleparallel gravity theory, despite its equivalence to GR, a rather peculiar theory.

One other possibility is to consider fermionic fields as gravitational sources for an expanding universe [5,6]. These fermionic sources have been investigated by using several approaches, including perturbations, anisotropy-to-isotropy scenarios, numerical and exact solutions, cyclic cosmologies and dark spinors. In this case the fermionic field plays the role of the inflaton in the early time of the Universe and of dark energy for the late time Universe, without the need of a scalar field or a cosmological constant term.



In the last years, the non-canonical scalar field called  $k$ -essence model has received much attention, where it is still worth investigating in a systematic way the possible cosmological behavior of the  $k$ -essence. Similarly to  $k$ -essence, a new model named as  $f$ -essence (non-canonical fermion field) was proposed [7]. In the present work the connection between teleparallel gravity and the  $f$ -essence is done via the tetrad formalism, where the components of the tetrad play the role of the gravitational degrees of freedom. Also recently, a new model named as  $g$ -essence was proposed in which is a more generalized version of  $k$ -essence and  $f$ -essence [8-11]. Note that  $f$ -essence is the fermionic counterpart of  $k$ -essence.

One of the most popular methods of finding the exact cosmological solutions is to use the Noether symmetry approach [12-15]. In addition, the existence of Noether symmetry leads to a specific form of the unknown functions that appear in the Lagrangian. The method is used to obtain cosmological models in several theory of gravity.

The structure of this paper is the following. In Sect. 2, the field equations are derived from a point-like Lagrangian for Friedman-Robertson-Walker (FRW) spacetime, which is obtained from an action including the fermionic field non-minimally coupled to the torsion scalar in the framework of teleparallel gravity. In Sect. 3, we search for the Noether symmetry of the Lagrangian of the theory and in Sect. 4, we obtain exact solutions of the field equations by using the coupling function and potential obtaining Noether symmetry approach. Finally, in the Sect. 5, we conclude with a brief summary of the obtained results. It should be noted that we fully adopt the natural system of units by taking  $8\pi G = c = \hbar = 1$ .

## 2. Action and field equations

Action for teleparallel gravity is non-minimally coupled with  $f$ -essence reads

$$S = \int d^4x e \{F(\Psi)T + 2K(\Psi, Y)\}, \quad (1)$$

where  $e = \det(e_\mu^a) = \sqrt{-g}$ ,  $e_\mu^a$  is a tetrad,  $T$  is a torsion scalar, and  $\psi$  and  $\bar{\psi} = \psi^\dagger \gamma^0$  denote the spinor field and its adjoint, with the dagger representing complex conjugation.  $F(\Psi)$  and  $V(\Psi)$  are generic functions, where  $\Psi = \bar{\psi}\psi$  which representing the coupling and the self-interaction potential of the fermionic field, respectively.

We will consider homogeneous and isotropic FRW metric which is given by

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2], \quad (2)$$

where  $a(t)$  is the scale factor of the universe.

The torsion scalar corresponding to the FRW metric (2) takes the form of  $T = -\frac{6\dot{a}^2}{a^2}$ , where the dot represents differentiation with respect to cosmic time  $t$ . Considering the background in Eq.(2), it is possible to obtain the point-like Lagrangian from the action (1)

$$L = 6Fa\dot{a}^2 - 2a^3K, \quad (3)$$

Equations of motion for  $f$ -essence part are obtained from the point-like Lagrangian (3)

$$K_Y \dot{\psi} + 0.5 \left( 3HK_Y + \dot{K}_Y \right) \psi - i\gamma^0 K' \psi + 3i\gamma^0 H^2 F' \psi = 0, \quad (4)$$

$$K_Y \dot{\bar{\psi}} + 0.5 \left( 3HK_Y + \dot{K}_Y \right) + iK' \bar{\psi} \gamma^0 - 3iH^2 F' \bar{\psi} \gamma^0 = 0, \quad (5)$$

where  $H = \dot{a}/a$  denotes the Hubble parameter and the prime denotes a derivative with respect to the bilinear  $\Psi$ . On the other hand, from the point-like Lagrangian (3) and by considering the Dirac equations, we find first Friedmann equation

$$3H^2 + 2\dot{H} + 2H\frac{\dot{F}}{F} + \frac{1}{F}p_f = 0. \quad (6)$$

Also we can find second Friedmann equation as follows

$$3H^2 - \frac{\rho_f}{F} = 0. \quad (7)$$

In the first and second Friedmann equations,  $\rho_f$  and  $p_f$  are the effective energy density and pressure of the fermion field, respectively, so that they have the following forms

$$\rho_f = K_Y Y - K, \quad (8)$$

$$p_f = K. \quad (9)$$

In expressions (4)-(9) is very hard to find solution since these are high order non-linear systems. In order to solve the field equations we have to determine a form for the coupling function and the potential density of the theory. To do this, in the following section we will use the Noether symmetry approach.

### 3. The Noether symmetry approach

Symmetries of the Lagrangian, the so-called Noether symmetry, can be used to obtain cosmological solutions. The Noether symmetry approach tells us that the Lie derivative of the Lagrangian with respect to a given vector field  $\mathbf{X}$  vanishes, i.e.

$$\mathcal{L}_{\mathbf{X}} L = 0. \quad (10)$$

If condition (10) satisfied, then  $\mathbf{X}$  is said to be a symmetry for the dynamics derived from the Lagrangian  $L$  and thus generates a conserved quantity. They determined the coupling and potential density of the fermionic field and showed that the fermionic field behaves as an inflaton describing an accelerated inflationary scenario. Now we will search for the Noether symmetries for our model. In terms of the components of the spinor field  $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$  and its adjoint  $\bar{\psi} = (\psi_1^\dagger, \psi_2^\dagger, -\psi_3^\dagger, -\psi_4^\dagger)$ , the Lagrangian (3) can be rewritten as

$$L = 6Fa\dot{a}^2 - 2a^3K. \quad (11)$$

Now we seek the condition for the Lagrangian (11) to admit a Noether symmetry. The existence of Noether symmetry given by (10), which implies the existence of a vector field  $\mathbf{X}$  such that

$$\mathbf{X} = \alpha \frac{\partial}{\partial a} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \sum_{j=1}^4 \left( \beta_j \frac{\partial}{\partial \psi_j} + \dot{\beta}_j \frac{\partial}{\partial \dot{\psi}_j} + \gamma_j \frac{\partial}{\partial \psi_j^\dagger} + \dot{\gamma}_j \frac{\partial}{\partial \dot{\psi}_j^\dagger} \right), \quad (12)$$

where  $\alpha, \beta_j$  and  $\gamma_j$  are unknown functions of the variables  $a, \psi_j$  and  $\psi_j^\dagger$ . Hence the Noether condition (10) leads to the following differential equations consisting of the coupled system of equations

$$\alpha + 2a \frac{\partial \alpha}{\partial a} + \frac{F'}{F} a \sum_{i=1}^4 (\epsilon_i \beta_i \psi_i^\dagger + \epsilon_i \gamma_i \psi_i) = 0, \quad (13)$$

$$12Fa \frac{\partial \alpha}{\partial \psi_j} = 0, \quad 12Fa \frac{\partial \alpha}{\partial \psi_j^\dagger} = 0, \quad (14)$$

$$3\alpha \psi_j + a \beta_j - a \sum_{i=1}^4 \left( \frac{\partial \beta_i}{\partial \psi_j^\dagger} \psi_i^\dagger - \frac{\partial \gamma_i}{\partial \psi_j} \psi_i \right) = 0, \quad (15)$$

$$3\alpha\psi_j^\dagger + a\gamma_j + a \sum_{i=1}^4 \left( \frac{\partial\beta_i}{\partial\psi_j} \psi_i^\dagger - \frac{\partial\gamma_i}{\partial\psi_j} \psi_i \right) = 0, \quad (16)$$

$$\sum_{i=1}^4 \left( \frac{\partial\beta_i}{\partial a} \psi_i^\dagger - \frac{\partial\gamma_i}{\partial a} \psi_i \right) = 0, \quad (17)$$

$$3\alpha(K - K_Y Y) + aK' \sum_{i=1}^4 (\epsilon_i \beta_i \psi_i^\dagger + \epsilon_i \gamma_i \psi_i) = 0, \quad (18)$$

where  $\epsilon_i = \begin{cases} 1 & \text{for } i = 1, 2 \\ -1 & \text{for } i = 3, 4 \end{cases}$ . This system given by Eqs.(13)-(18), is obtained by imposing the fact that the coefficients of  $\dot{a}^2, \dot{a}, \dot{\psi}_j, \dot{\psi}_j^\dagger, \dot{a}\dot{\psi}_j$  and  $\dot{a}\dot{\psi}_j^\dagger$  vanish.

One can see from Eq.(14) that the coefficient  $\alpha$  is a function only depends on  $a$ . From the Eq.(18) one can rewrite as follows:

$$\frac{3\alpha(K - K_Y Y)}{aK'} = - \sum_{i=1}^4 (\epsilon_i \beta_i \psi_i^\dagger + \epsilon_i \gamma_i \psi_i). \quad (19)$$

We put (19) into (13) and, recalling that  $F$  and  $V$  are only functions of  $\Psi$ , the corresponding result is

$$\frac{\alpha}{a} \frac{\partial\alpha}{\partial a} = \frac{3F'(K - K_Y Y)}{2FK'} - \frac{1}{2} = n, \quad (20)$$

where  $n$  is a constant. Then, we find  $\alpha$  from (20)

$$\alpha = \alpha_0 a^n, \quad (21)$$

where  $\alpha_0$  is an integration constant. Now, from (15), (16) and (21), after some algebraic calculations, one can obtain the solutions for the other symmetry generators  $\beta_j$  and  $\gamma_j$  as follows:

$$\begin{aligned} \beta_j &= -\left(\frac{3}{2}\alpha_0 a^{n-1} + \epsilon_j \beta_0\right)\psi_j, \\ \gamma_j &= -\left(\frac{3}{2}\alpha_0 a^{n-1} - \epsilon_j \beta_0\right)\psi_j^\dagger, \end{aligned} \quad (22)$$

where  $\beta_0$  is a constant of integration. Using the above solution in (20), the coupling function  $F(\Psi)$  is obtained

$$\frac{K - YK_Y}{K'} = \left(\frac{2n+1}{3}\right) \frac{F}{F'} = \Psi, \quad (23)$$

$$\Psi \frac{dF}{d\Psi} - \left(\frac{2n+1}{3}\right) F = 0; \quad (24)$$

$$F(\Psi) = C_1 \Psi^{\frac{2n+1}{3}}, \quad (25)$$

where  $C_1$  is an integrable constant.

Substituting these values into equation (20) and using the  $\alpha$  given in (25), we have the following equation

$$Y \frac{\partial K}{\partial Y} + \Psi \frac{\partial K}{\partial \Psi} - K = 0, \quad (26)$$

To solve this equation we need to make the change of variables

$$Y, \Psi \rightarrow p = Y - \nu\Psi, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial p}, \quad \frac{\partial}{\partial \Psi} = -\nu \frac{\partial}{\partial p}. \quad (27)$$

where  $\nu$  is a constant. Then the partial differential equation (26) is transformed into an ordinary differential equation

$$K - p \frac{dK}{dp} = 0, \quad (28)$$

Then, we find  $K$  as

$$K(\Psi, Y) = \mu Y - \nu\Psi, \quad (29)$$

where  $\mu, \nu$  are some integral constants. For our model we have  $V = \nu\Psi$ ,

#### 4. Exact cosmological solutions

In this section, we attempt to integrate the dynamical system given by (4)-(5) analytically. Since the coupling and potential functions depend on the bilinear function  $\Psi$ , using the Dirac equations (4) and (5) one gets

$$\dot{\Psi} + 3\frac{\dot{a}}{a}\Psi = 0, \quad (30)$$

and integration gives

$$\Psi = \frac{\Psi_0}{a^3}, \quad (31)$$

where  $\Psi_0$  is a constant of integration. We note that, since the field equations can be directly integrable, it is not necessary to calculate the constants of motion associated with the Noether symmetry. Also the constants of motion give no new constraint on the field equations. From the above solution, the first and second Friedmann equations become only a function of the cosmic scale factor

$$\dot{a} = \frac{\nu^{\frac{1}{2}} \Psi_0^{\frac{1}{3}(1-n)}}{\sqrt{3C_1}} a^n. \quad (32)$$

or

$$\dot{a} = a_0 a^n. \quad (33)$$

where  $a_0$  is a constant

$$a_0 = \frac{\nu^{\frac{1}{2}} \Psi_0^{\frac{1}{3}(1-n)}}{\sqrt{3C_1}}. \quad (34)$$

If we solve this equation, we have solution

$$a(t) = [a_0 (1-n)t + C_1]^{-\frac{1}{n-1}}, \quad (35)$$

where  $C_1$  is integral constant.

The Hubble parameter can be written as

$$H = \frac{a_0}{a_0 (1-n)t + C_1}. \quad (36)$$

Defining the energy density and pressure

$$\rho = \frac{3a_0}{[(1-n)t + C_1]^2}, \quad (37)$$

$$p = -\frac{(2n+1)a_0^2}{[(1-n)t + C_1]^2}. \quad (38)$$

The equation of state  $\omega$  be written as

$$\omega = -\frac{1}{3}(2n+1), \quad (39)$$

And the deceleration parameter  $q$  is defined by:

$$q = \frac{1}{2}(1+3\omega) = -n \quad (40)$$

Cosmological observations denote that  $w$  lies in a very narrow strip close to  $w = -1$ . The case  $w = -1$  corresponds to the cosmological constant. For  $w < -1$ , the phantom phase is observed, and for  $-1 < w < -1/3$  the phase is described by quintessence. Thus, in the interval  $0 < n < 1$ , we have the quintessence phase. If  $n > 1$ , then the phantom phase occurs, where the universe is both expanding and accelerating. Therefore, we conclude that the fermionic field behaves as both the quintessence and phantom dark energy.

## 5. Conclusions

In the present work, we have investigated the  $f$ -essence dark energy models, where the gravitational part of the Lagrangian is considered teleparallel gravity. We know that Noether symmetry a very important tool, because it guarantees the conservation laws and restricts the possible expressions for the coupling function and for the potential of fermion field in the framework of teleparallel gravity. In our case, this symmetry yields cosmological solutions that describe not only the early-time but also late-time accelerated expansion.

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